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THE ŁOJASIEWICZ EXPONENT FOR WEIGHTED HOMOGENEOUS POLYNOMIALS OF TWO REAL VARIABLES

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Abstract

The purpose of this paper is to give the exact value of the Łojasiewicz exponent for an isolated weighted homogeneous polynomials of two real variables in terms of its weights.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function. The Łojasiewicz exponent $L(f)$ of f is by definition

$$L(f) := \inf\{\lambda > 0 : |\operatorname{grad} f| \geq \text{const.}|x|^{\lambda} \text{near zero}\}.$$

It is well known (see [3, 8]) that the Łojasiewicz exponent can be calculated by means of analytic paths

$$L(f) = \sup\left\{\frac{\operatorname{ord}(\operatorname{grad} f(\varphi(t)))}{\operatorname{ord}(\varphi(t))} : 0 \neq \varphi(t) \in \mathbb{R}(t)^n,\ \varphi(0) = 0\right\}, \tag{0.1}$$

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where $\operatorname{ord}(\phi) := \inf_i\{\operatorname{ord}(\phi_i)\}$ for $\varphi \in \mathbb{R}(t)^n$. By definition, we put $\operatorname{ord}(0) = +\infty$. It is also known that $L(f) < +\infty$ if and only if f has an isolated singularity at the origin. The computation or estimation of the Łojasiewicz exponent is a quite interesting problem not only in geometric analysis but also in singularity theory. For example, Kuiper-Kuo theorem [6, 7] proved that for any integer r greater than $L(f)$, f is a C^0- sufficient, r- jet, i.e., adding to the function f monomials of order greater than $L(f)$ does not change its topological type. Bochnak and Łojasiewicz [2] showed that C^0- sufficiency degree of f (i.e., the minimal integer r such that f is C^0- sufficient, r- jet) is equal to $[L(f)] + 1$, where $[L(f)]$ denotes integral part of $L(f)$.

Observation. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be weighted homogeneous polynomials with isolated singularities of degree d, and let $w = (w_1, \ldots, w_n)$ be the weights of f, i.e., f is weighted homogeneous of type $(d; w)$, recently the author in [1] and Brzostowski [4] proved that the Łojasiewicz exponent of f is precisely equal to

$$L(f) = \max_{i=1}^{n}\left(\frac{d}{w_i} - 1\right).$$

Estimates of the Łojasiewicz exponent for weighted homogeneous isolated singularities in the real cases are in a recent paper by Haraux and Pham [5].

Motivated by the above observation, we are looking to establish the Łojasiewicz exponent for the classes of weighted homogeneous polynomials of two real variables in terms of the weights. To prove the main result (Theorem 3 below), we recall the notion of weighted homogenous filtration introduced by Paunescu in [9]. By using it and the generalized Euler identity, we can compute the Łojasiewicz exponent of weighted homogeneous polynomials of two real variables.

Notation. To simplify the notation, we will adopt the following conventions: for a function $g(x, y)$ we denote by ∂g the gradient of g and by $\partial_x g$ the gradient of g with respect to variables x.

Let $\varphi, \psi : (\mathbb{R}^n, 0) \to \mathbb{R}$ be two function germs. We say that $\varphi(x) \lesssim \psi(x)$ if there exists a positive constant $C > 0$ and an open neighborhood U of the origin in $\mathbb{R}^n$ such that $\varphi(x) \leq C\psi(x)$, for all $x \in U$. We write $\varphi(x) \sim \psi(x)$ if $\varphi(x) \lesssim \psi(x)$ and $\psi(x) \lesssim \varphi(x)$. Finally, $|\varphi(x)| \ll |\psi(x)|$ (when x tends to x_0) means

$$\lim_{x \to x_0} \frac{\varphi(x)}{\psi(x)} = 0.$$

1. Weighted Homogeneous Filtration, Main Results

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathcal{O}_n$ denote the ring of analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$.

From now, we shall fix a system of positive integers $w = (w_1, \ldots, w_n) \in \mathbb{N} - \{0\})^n$, the weights of variables x_i, $w(x_i) = w_i$, $1 \leq i \leq n$, and a positive integer d, then a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is called *weighted homogeneous of degree* d with respect the weight $w = (w_1, \ldots, w_n)$ (or type $(d; w)$) if f may be written as a sum of monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with

$$\alpha_1 w_1 + \cdots + \alpha_n w_n = d. \tag{1.1}$$

We say that an analytic function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is non-degenerate if $\left\{\frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\right\} = \{0\}$ as germs at the origin of $\mathbb{R}^n$.

We may introduce (see [9]) the function $\rho(x) = \left(|x_1|^{\frac{2}{w_1}} + \cdots + |x_n|^{\frac{2}{w_2}}\right)^{\frac{1}{2}}$. We also consider the spheres associated to this ρ,

$$S_r = \{x \in \mathbb{R}^n : \rho(x) = r\}, \quad r > 0.$$

Here $\cdot$ means the weighted action, with respect to the $\mathbb{R}^*$ action defined below

$$t \cdot x = (t^{w_1} x_1, \ldots, t^{w_n} x_n).$$

Definition 1. Using ρ, we define a singular Riemannian metric on $\mathbb{R}^n$ by the following bilinear form

$$\left\langle \rho^{w_i} \frac{\partial}{\partial x_i}, \rho^{w_j} \frac{\partial}{\partial x_j} \right\rangle = \delta_{i,\, j} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We will denote by grad_w and $\| \ \|_w$, the corresponding gradient and norm associated with this Riemannian metric (for more details about these see [9]).

Let $f \in \mathcal{O}_n$. We denote the Taylor expansion of f at the origin by $\sum c_v x^v$. Setting $H_j(x) = \sum c_v x^v$, where the sum is taken over n with $< w, v > = w_1 v_1 + \cdots + w_n n_n = j$, we can write the weighted homogeneous Taylor expansion f,

$$f(x) = H_d(x) + H_{d+1}(x) + \cdots; \quad H_d \neq 0.$$

We call d the weighted degree of f and H_d the weighted initial form of f about the weight. Furthermore, for any $f \in \mathcal{O}_n$, we get

$$\| \operatorname{grad}_w f(x) \|_w \lesssim \rho^{d_w(f)}(x), \tag{1.2}$$

where $d_w(f)$ denotes the degree of f with respect to w. Indeed, as all nonzero x, we find $\frac{1}{\rho(x)} \cdot x \in S_1$, moreover, we have $\frac{\partial H_j}{\partial x_i}$ is zero or a weighted homogeneous polynomial of degree $d - w_j$, then we obtain

$$\begin{aligned} \left\| \operatorname{grad}_w H_j\left(\frac{1}{\rho(x)} \cdot x\right) \right\|_w &= \frac{\| \operatorname{grad}_w H_j(x) \|_w}{\rho(x)^j} \\ &\lesssim 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \| \operatorname{grad}_w f(x) \|_w &\lesssim \sum_{j \geq d_w(f)} \| \operatorname{grad}_w H_j(x) \|_w \\ &\lesssim \rho^{d_w(f)}(x). \end{aligned}$$

Proposition 2. *Let* $f \in \mathcal{O}_n$ *be a weighted homogeneous isolated singularity of type* $(d;\, w)$ *at* $0 \in \mathbb{R}^n$. *Then*

$$\| \operatorname{grad}_w f(x) \|_w \gtrsim \rho(x)^d. \tag{1.3}$$

Proof. Since f has only isolated singularity at the origin, for small values of τ, we have

$$\begin{aligned} \| \operatorname{grad}_w f(x) \|_w &= \left(\sum_{i=1}^{n} \left| \rho^{w_i}(x) \frac{\partial f}{\partial z_i}(z) \right|^2 \right)^{\frac{1}{2}} \\ &\gtrsim 1, \quad \forall x \in S_r. \end{aligned} \tag{1.4}$$

On the other hand, $\dfrac{\partial f}{\partial x_i}$ is weighted homogeneous of degree $d - w_i$ for $i = 1, \ldots, n$ and also, $\dfrac{r}{\rho(x)} \cdot x \in S_r$ for all nonzero x. Thus, by (1.4) we obtain

$$\begin{aligned} \left\| \operatorname{grad}_w f\left(\frac{r}{\rho(x)} \cdot x \right) \right\|_w &= r^d \frac{\| \operatorname{grad}_w f(x) \|_w}{\rho(x)^d} \\ &\gtrsim 1. \end{aligned}$$

This completes the proof of the proposition. □

The main result of this paper is the following:

Theorem 3. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be non-degenerate weighted homogeneous polynomial of type* $(d;\, w_1,\, w_2)$ *such that* $w_1 \geq w_2$. *Then*

$$L(f) = \begin{cases} \dfrac{d - w_1}{w_2}, & \text{if } \left\{ \dfrac{\partial f}{\partial x_1}(x_1,\, x_2) = 0 \right\} \subset \{x_2 = 0\}, \\ \dfrac{d}{w_2} - 1, & \text{if not}. \end{cases}$$

2. Proof of Theorem 3

We first note that in the case where $w_1 = w_2$ (i.e., homogenous filtration), so we

can find from the (1.2) and (1.3) that

$$\| \operatorname{grad}_w f(x) \|_w \sim \rho(x)^d.$$

But $\| \operatorname{grad}_w f(x) \|_w \sim \| x \| \| \partial f \|$ and $\rho(x)^d \sim \| x \|^{\frac{d}{w_1}} = \| x \|^{\frac{d}{w_2}}$. Hence, we get

$$\begin{aligned} L(f) &= \frac{d}{w_2} - 1 \\ &= \frac{d - w_1}{w_2}. \end{aligned}$$

From now, we suppose that $w_1 > w_2$. There are two cases to be considered.

Case 1. In this case, we suppose that

$$\left\{ \frac{\partial f}{\partial x_1}(x_1, x_2) = 0 \right\} \subset \{x_2 = 0\},$$

take an analytic

$$\begin{aligned} \varphi(t) &= (t^{w_1}, t^{w_2}) \\ &= t \cdot (1, 1), \end{aligned}$$

then from (0.1) we get

$$\begin{aligned} L(f) &\geq \frac{\operatorname{ord}(\partial f(\varphi(t)))}{\operatorname{ord}(\varphi(t))} \\ &= \frac{d - w_1}{w_2}. \end{aligned}$$

Since f defining an isolated singularity at the origin $0 \in \mathbb{R}^n$, there exist the terms $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ with the origin is an isolated zero of $\partial_x \varphi$, i.e., $\{\partial_{x_1} \varphi = 0\} = \{0\}$ and

$$f(x_1, x_2) = \varphi(x_1, x_2) \quad \text{or} \quad f(x_1, x_2) = x_2 \varphi(x_1, x_2).$$

For the case where the origin is an isolated zero of $\partial_{x_1} f$, since $\partial_{x_1} f$ w-form of

degree $d - w_1$, it follow from (1.4) that

$$\begin{aligned} \| \partial f \| \gtrsim \rho^{d-w_1} &\gtrsim (| x_1 |^{1/w_1} + | x_2 |^{1/w_2})^{d-w_1} \\ &\gtrsim \| (x_1, x_2) \|^{\frac{d-w_1}{w_2}}. \end{aligned}$$

Hence $L(f) \leq \dfrac{d - w_1}{w_2}$.

This ends the proof of Theorem 3 in the case where the origin is an isolated zero of $\partial_{x_1} f$.

From now we suppose that $f(x_1, x_2) = x_2 \varphi(x_1, x_2)$, it is easy to see that φ is weighted homogeneous of degree $d - w_2$ (type $(d - w_2; w_1, w_2)$), we have that

$$\| \partial f \|^2 = | x_2 \partial_{x_1} \varphi |^2 + | \varphi + x_2 \partial_{x_2} \varphi |^2.$$

Moreover, it follows from the generalized Euler identity that

$$\varphi(x_1, x_2) = \frac{w_1}{d - w_2} x_1 \partial_{x_1} \varphi(x_1, x_2) + \frac{w_2}{d - w_2} x_2 \partial_{x_2} \varphi(x_1, x_2).$$

Then, we obtain that

$$\begin{aligned} \| \partial f \|^2 = \left\| \left(\frac{w_1}{d - w_2} x_1, x_2 \right) \right\|^2 | \partial_{x_1} \varphi |^2 + 2 \frac{w_1 d}{d - w_2} x_1 x_2 \partial_{x_1} \varphi \partial_{x_2} \varphi \\ + \left| \frac{d}{d - w_2} x_2 \partial_{x_2} \varphi \right|^2. \end{aligned}$$

But, it follows from $w_1 > w_2$ that

$$\begin{aligned} d_w((\partial_{x_1} \varphi)^2) &= 2(d - w_2 - w_1) \\ &< d - w_2 - w_1 + d - w_2 - w_2 \\ &= d_w(\partial_{x_1} \varphi \partial_{x_2} \varphi) \\ &< d_w((\partial_{x_2} \varphi)^2). \end{aligned}$$

Therefore, by the origin is an isolated zero of $\partial_{x_1}\varphi$ we get that

$$\begin{aligned}\| \partial f \| &\gtrsim \| (x_1, x_2) \| \rho^{d-w_2-w_1} \\ &= \| (x_1, x_2) \| (| x_1 |^{1/w_1} + | x_2 |^{1/w_2})^{d-w_2-w_1} \\ &\gtrsim \| (x_1, x_2) \|^{\frac{d-w_1}{w_2}}.\end{aligned}$$

Hence $L(f) \leq \dfrac{d - w_1}{w_2}$.

This ends the proof of Theorem 3 in the first case.

Case 2. In this case, we suppose that

$$\left\{\frac{\partial f}{\partial x_1}(x_1, x_2) = 0\right\} \not\subseteq \{x_2 = 0\}.$$

Let $a = (a_1, a_2) \in \left\{\dfrac{\partial f}{\partial x_1}(x_1, x_2) = 0\right\}$ and $a_2 \neq 0$, take an analytic path

$$\begin{aligned}\varphi(t) &= (t^{w_1} a_1, t^{w_2} 2_2) \\ &= t \cdot a.\end{aligned}$$

Then from (0.1), we get

$$\begin{aligned}L(f) &\geq \frac{\mathrm{ord}(\partial f(\varphi(t)))}{\mathrm{ord}(\varphi(t))} \\ &\geq \frac{d - w_2}{w_2}.\end{aligned}$$

On the other hand, by Proposition 2, we obtain that

$$\begin{aligned}\rho^{w_2} \| \partial f \| &\gtrsim \| \mathrm{grad}_w f(x) \|_w \\ &\gtrsim \rho(x)^d.\end{aligned}$$

Then

$$\begin{aligned}\| \partial f \| &\gtrsim \rho^{d-w_2} \\ &= (| x_1 |^{1/w_1} + | x_2 |^{1/w_2})^{d-w_2} \\ &\gtrsim \| (x_1, x_2) \|^{\frac{d-w_2}{w_2}}.\end{aligned}$$

Hence $L(f) \leq \dfrac{d - w_2}{w_2}$.

This ends the proof of Theorem 3.

We conclude with several examples.

Example 4. Let

$$f(x, y) = x^3 + xy^6 + y^9,$$

f is weighted homogenous of type $(9; 3, 1)$ defining an isolated singularity, since $\{\partial_x f = 0\} = \{0\}$, by Theorem 3. We get $L(f) = 6$. Also, for $g(x, y) = x^3 - xy^6 + y^9$ can be seen as weighted homogenous of the same type, but $\{\partial_x g = 0\} \not\subseteq \{y = 0\}$, hence by Theorem 3 we get $L(g) = 8$.

Example 5. Let

$$f(x, y) = y(x^5 + xy^{12} + y^{15}),$$

f is weighted homogenous of type $(16; 3, 1)$ defining an isolated singularity, since $\{\partial_x f = 0\} \subset \{y = 0\}$, then by Theorem 3 we get $L(f) = 13$. Also, for $g(x, y) = y(x^5 - xy^{12} + y^{15})$ can be seen as weighted homogenous of the same type, but $\{\partial_x g = 0\} \not\subseteq \{y = 0\}$, so by Theorem 3 we obtain $L(g) = 15$.

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