JP Journal of Applied Mathematics
Volume 15, Issue 2, 2017, Pages 63-79

ASSET ALLOCATION UNDER A LOGARITHMIC UTILITY FUNCTION WITH REGIME SWITCHING

FENGXIA HU

School of Statistics and Mathematics
Shanghai Lixin University of Accounting and Finance
Shanghai, 201209
P. R. China

Abstract

Under a logarithmic utility function, this paper studies optimal portfolio problem in a discrete-time, finite-horizon setting, where short-selling and leveraging are prohibited. We suppose that the random return of risky asset depends on the economic environments which are described by a Markov chain. Employing dynamic programming theory, we obtain a closed form solution of the optimal investment strategy. In addition, with the help of stochastic orders, we discuss the properties of the optimal investment strategy, and then investigate the impact of economic environment regimes on the optimal strategy. Finally, we derive the order of ranking the optimal proportions invested in risky asset.

1. Introduction

Asset allocation problem is of great importance in finance from both theoretical and practical perspectives. The pioneering work of Markowitz [11] first provided a mathematically elegant way to formulate the optimal portfolio allocation problem,

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2010 Mathematics Subject Classification: 91G10, 90C39, 60J10.

Keywords and phrases: Stochastic orders, Markov chain, optimal investment strategy, logarithmic utility function, dynamic programming.

Received July 29, 2017; Revised September 19, 2017

and it developed the well-known mean-variance approach for optimal portfolio allocation. The author considered a single period model and adopted variance (or standard deviation) as a measure of risk in the portfolio. The novelty of this approach is that it reduces the optimal portfolio allocation problem to the one, where only mean and variance of the rates of returns of the risky assets are involved under the normality assumption for these rates. This greatly simplifies the problem of optimal portfolio allocation and makes a great leap forward in the development of the field. Merton [12, 13] studied the optimal portfolio allocation problem in a continuous-time framework, which provided a more realistic setting to deal with the problem. He employed stochastic optimal control techniques to provide an elegant solution to the optimal portfolio allocation problem. His work opens up an important field in modern finance, namely, the continuous-time finance. Under the assumption that the returns from the risky assets were stationary, Merton derived closed-form solutions to the optimal portfolio allocation in a continuous-time setting. In reality, the returns from the risky assets sometimes might not be stationary. Boyle and Yang [1] considered the optimal asset allocation problem in the presence of non-stationary asset returns and transaction costs. They concentrated on the Duffie and Kan [3] multi-factor stochastic interest model and adopted a viscosity solution approach to deal with the problem. Recently, regime-switching, or Markov-modulated, models have received much attention among both researchers and market practitioners. Hamilton [10] pioneered the econometric applications of regime-switching models by considering a discrete-time Markov-switching autoregressive time series model. Since then, regime-switching models, both discrete-time and continuous-time, have found a wide range of applications in economics and finance. Some papers with regime-switching models in finance include: Elliott and van der Hoek [6] and Cheung and Yang [2] for asset allocation, Pliska [14] and Elliott et al. [8], Elliott and Kopp [5] for short rate models, Elliott and Hinz [4] for portfolio analysis and chart analysis, Elliott et al. [7] and Guo [9] for option pricing under incomplete markets. Furthermore, regime-switching models provide a convenient way to describe the impact of the structural changes in economic conditions and business cycles on the price dynamics. They also provide a pertinent way to describe the non-stationary feature of returns of risky assets. More recently, Yin and Zhou [17] and Zhou and Yin [18] established a mean-variance portfolio selection problem under Markovian regime-switching models in a continuous-time economy. They introduced the stochastic linear quadratic control to deal with the problem, and then established closed form solutions to mean-variance efficient portfolios and efficient frontiers.

With the bankruptcy constraint, Sotomayor and Cadenillas [16] considered an optimal consumption and investment problem under a Markovian regime-switching model for the asset price dynamics. They determined a consumption-investment policy so as to maximize the expected total discounted utility of consumption until bankruptcy, and they employed techniques of classical stochastic optimal control to derive the regime-switching Hamilton-Jacobi-Bellman equation.

The rest of this paper is organized as follows. Section 2 gives preliminary and models studied in this paper. In Section 3, a closed form solution of the optimal investment strategy is obtained. Finally, in Section 4, we discuss the properties of the optimal investment strategy, and then investigate the impact of economic environment regimes on the optimal strategy. At last, we derive the order of ranking the optimal proportions invested in risky asset.

2. Preliminary and Models

To begin with, we point out that all random variables in this paper are defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

We will study the optimal allocation problem in a discrete-time, finite-horizon setting. Throughout this paper, the investment horizon $T \in \mathbb{N}$ is fixed. We assume that there are two assets being traded in the market: a risk free asset (say, bank bond) and a risky asset (say, stocks). The risk free asset will earn a deterministic return $R > 1$ over each single time period. Suppose that $\{\xi_n, n = 0, 1, 2, \ldots, T\}$ is a discrete-time and time-homogeneous Markov chain. The random return of the risky asset over different time periods will depend on state of the chain $\{\xi_n, n = 0, 1, 2, \ldots, T\}$ at the beginning of that time period. Different states of the Markov chain represent different investment environments of the risky asset. We further assume that the state space of this Markov chain is $Z = \{1, 2, \ldots, s\}$, the transition probability matrix is denoted as $P = \{p_{ij}\}$. The wealth of an investor at time n will be denoted as W_n, and the random return in time period $[n, n+1]$ given that $\xi_n = i \in Z$ is denoted as $R_n^i > 1$. Then a proportion α_n of W_n will be invested in the risky asset with $0 \leq \alpha_n \leq 1$, and the rest will be invested in the risk-free asset. The constraints $0 \leq \alpha_n \leq 1$, $n = 0, 1, 2, \ldots, T$, mean that short selling and leveraging are prohibited, and they are able to avoid the possibility that the wealth becomes negative. And the sequence $\{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{T-1}\}$ which satisfies the constraints mentioned above is called *investment strategy*.

The wealth of the investor will evolve according to

$$W_{n+1} = W_n[\alpha_n R_n^{\xi_n} + (1-\alpha_n)R],$$

for $n = 0, 1, 2, \ldots, T-1$.

We assume that the utility function takes the following form

$$U(x) = \ln x,$$

which is called a *logarithmic utility function*.

Given that the investor has an initial endowment w of wealth W_0 and the Markov chain is initially at regime $i \in Z$, the objective of the investor is to maximize the utility of the terminal discounted wealth:

$$\max_{\{\alpha_0, \alpha_1, \ldots, \alpha_{T-1}\}} \mathbb{E}\left[\ln\left(\frac{W_T}{(1+\rho)^T}\right) | \xi_0 = i, W_0 = w\right], \tag{2.1}$$

where $w > 0$, $i \in Z$, ρ is the discount rate. The optimal proportions that solve problem (2.1) is called the *optimal investment strategy*, and is denoted as

$$\{\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_{T-1}\}.$$

The following assumptions are made throughout this paper:

(1) for fixed $i \in Z$, the random returns $R_0^i, R_1^i, \ldots, R_{T-1}^i$ are identically distributed with common distribution function $F_i(\cdot)$, having support in $[0, +\infty)$ and are integrable;

(2) in different time periods, the random returns are independent, i.e., $\forall i, j \in Z$, R_n^i is independent of R_m^j for $m \neq n$.

(3) the Markov chain $\{\xi_n, n = 0, 1, 2, \ldots, T\}$ is independent of all random returns:

$$\mathbb{P}(\xi_{n+1} = i_{n+1}, R_n^{i_n} \in B | \xi_0 = i_0, \ldots, \xi_n = i_n) = p_{i_n i_{n+1}} \mathbb{P}(R_n^{i_n} \in B)$$

for all $i_0, \ldots, i_n, i_{n+1} \in Z$, $B \in \mathfrak{B}(\mathbb{R})$ and $n = 0, 1, \ldots, T-1$, where $\mathfrak{B}(\mathbb{R})$ is the Borel σ-field.

In order to obtain the optimal investment strategy, we first introduce an auxiliary function, and then discuss its properties. They turn out to be useful in studying our dynamic maximization problem.

Definition 2.1. Fix any $i \in Z$, and assume that R^i is a random variable with distribution function $F_i(\cdot)$. Define function $\tilde{M}_i(\cdot) : [0, 1] \to \mathbb{R}$ by

$$\tilde{M}_i(\alpha) = \mathbb{E}[\ln(\alpha R^i + (1-\alpha)R)].$$

The next proposition summarizes some basic properties of the function $\tilde{M}_i(\cdot)$. Further properties will be explored later.

Proposition 2.1. *For fixed* $i \in Z$, *the function* $\tilde{M}_i(\cdot)$ *is*

1. *well-defined on* $[0, 1]$ *under an extra condition that* $\ln R^i$ *is integrable*, *i.e.*, *for any* $\alpha \in [0, 1]$, *the random variable* $\ln(\alpha R^i + (1-\alpha)R)$ *is integrable under the condition that* $\ln R^i$ *is integrable*;

2. *is non-negative*;

3. *strictly concave on* $[0, 1]$;

4. *continuous on* $[0, 1]$.

Proof. 1. For any $\alpha : 0 \leq \alpha \leq 1$. When $\alpha = 1$, $\ln(\alpha R^i + (1-\alpha)R) = \ln R^i$; when $0 \leq \alpha < 1$, $\ln(1-a) < \ln(\alpha R^i + (1-\alpha)R) \leq \ln(R^i + R)$. Hence, by the assumption that $\ln(R^i)$ is integrable, the conclusion is true.

2. The conclusion is obviously true because both R and R^i are more than 1.

3. Choose arbitrarily $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 \neq \alpha_2$, and $0 < \beta < 1$, we have

$$\begin{aligned} &\tilde{M}_i(\beta\alpha_1 + (1-\beta)\alpha_2) \\ &= \mathbb{E}\{\ln[(\beta\alpha_1 + (1-\beta)\alpha_2)R^i + (1 - \beta\alpha_1 - (1-\beta)\alpha_2)R]\} \end{aligned}$$

$$= \mathbb{E}\{\ln[(\beta\alpha_1 + (1-\beta)\alpha_2)(R^i - R) + R]\}$$

$$= \mathbb{E}\{\ln[\beta(\alpha_1(R^i - R) + R) + (1-\beta)(\alpha_2(R^i - R) + R)]\}$$

$$> \mathbb{E}[\beta\ln(\alpha_1(R^i - R) + R) + (1-\beta)\ln(\alpha_2(R^i - R) + R)]$$

$$= \beta\mathbb{E}[\ln(\alpha_1 R^i + (1-\alpha_1)R)]$$

$$+ (1-\beta)\mathbb{E}[\ln(\alpha_2 R^i + (1-\alpha_2)R)]$$

$$= \beta\widetilde{M}_i(\alpha_1) + (1-\beta)\widetilde{M}_i(\alpha_2),$$

where the strict inequality follows from the fact that function $\ln x$ is strict concave on $(0, +\infty)$. This shows the strict concavity of $\widetilde{M}_i(\cdot)$.

4. From the proof of part 1, we know that the collection of random variables $\{\ln(\alpha R^i + (1-\alpha)R)\}_{0\le\alpha\le 1}$ is dominated by an integrable random variable. By Dominated Convergence Theorem, continuity of $\widetilde{M}_i(\cdot)$ follows.

Remark 2.1. By the above proposition, hereafter, we will always make the extra assumption that $\ln R^i$ is integrable. As a consequence of Proposition 2.1, $\widetilde{M}_i(\cdot)$ achieves its maximum on $[0, 1]$ at a unique point, which is denoted as α_i^*. The corresponding maximum value $\widetilde{M}_i(\alpha_i^*)$ is denoted as $M_i^{(1)}$.

3. Optimal Investment Strategy

With enough preparation above, now we can return to our problem. In order to employ the dynamic programming technique, it is necessary to introduce the following concept.

Definition 3.1. The value function $V_n : Z \times \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$V_n(i, w) = \begin{cases} \max_{\{\alpha_n, \alpha_{n+1}, \dots, \alpha_{T+1}\}} \mathbb{E}\left[\ln\left(\frac{W_T}{(1+\rho)^{T-n}}\right) \mid \xi_n = i, W_n = w\right], & \text{if } n = 0, 1, \dots, T-1, \\ \ln w, & \text{if } n = T, \end{cases}$$

where $w > 0$, $i \in Z$. Our objective is to compute $V_0(i, w)$, and then find the associated optimal investment strategy. By the theory of Dynamic programming, the value function and the optimal investment strategy can be obtained by solving the following Bellman Equation:

$$V_n(i, w) = \begin{cases} \max_{0 \le \alpha_n \le 1} \mathbb{E}\left[\frac{1}{1+\rho} V_{n+1}(\xi_{n+1}, W_{n+1}) \mid \xi_n = i, W_n = w\right], \\ \qquad\qquad\qquad\qquad \text{if } n = 0, 1, \ldots, T-1, \\ \ln w, \qquad \text{if } n = T, \end{cases}$$

where $w > 0$, $i \in Z$.

By using the important foundation above, we can establish the main result in the subsequent part, which explores the optimal investment strategy.

Theorem 3.1. *For* $n = 0, 1, \ldots, T$, *the value functions are given by*

$$V_n(i, w) = \begin{cases} \frac{1}{(1+\rho)^{T-n}}\left[\ln w + M_i^{(T-n)}\right], & \text{if } n = 0, 1, \ldots, T-1, \\ \ln w, & \text{if } n = T; \end{cases} \tag{3.1}$$

for any $w > 0$, $i \in Z$, *where* $M_i^{(\cdot)}$ *is defined by*

$$M_i^{(n+1)} = M_i^{(1)} + \sum_{j=1}^{s} p_{ij} M_j^{(n)}, \quad n = 1, 2, \ldots, T-1.$$

Moreover, the optimal investment strategy, which solves the Bellman Equation, is given by

$$\hat{\alpha}_n(i, w) = \alpha_i^*, \quad n = 0, 1, \ldots, T-1.$$

Proof. We prove the conclusion by induction. The result is obviously true when $n = T$. When $n = T-1$, we have

$$V_{T-1}(i, w)$$

$$= \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}\left[\frac{1}{1+\rho} V_T(\xi_T, W_T) \mid \xi_{T-1} = i, W_{T-1} = w\right]$$

$$= \frac{1}{1+\rho} \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln W_T \mid \xi_{T-1} = i, W_{T-1} = w]$$

$$= \frac{1}{1+\rho}$$

$$\times \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln(W_{T-1}(\alpha_{T-1} R_{T-1}^{\xi_{T-1}} + (1-\alpha_{T-1})R)) \mid \xi_{T-1} = i, W_{T-1} = w]$$

$$= \frac{1}{1+\rho} \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln(w(\alpha_{T-1} R_{T-1}^{i} + (1-\alpha_{T-1})R))]$$

$$= \frac{1}{1+\rho} \{\ln w + \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln(\alpha_{T-1} R_{T-1}^{i} + (1-\alpha_{T-1})R)]\}$$

$$= \frac{1}{1+\rho} [\ln w + \max_{0 \le \alpha_{T-1} \le 1} \tilde{M}_i(\alpha_{T-1})]$$

$$= \frac{1}{1+\rho} (\ln w + M_i^{(1)}).$$

So the conclusion is true when $n = T - 1$. Now we assume that the results are true when $n = k + 1$ for some $n = 0, 1, \dots, T - 1$, then when $n = k$,

$$V_k(i, w)$$

$$= \max_{0 \le \alpha_k \le 1} \mathbb{E}\left[\frac{1}{1+\rho} V_{k+1}(\xi_{k+1}, W_{k+1}) \mid \xi_k = i, W_k = w\right]$$

$$= \max_{0 \le \alpha_k \le 1} \mathbb{E}\left\{\frac{1}{(1+\rho)^{T-k}} [\ln W_{k+1} + M_{\xi_{k+1}}^{(T-k-1)} \mid \xi_k = i, W_k = w]\right\}$$

$$= \frac{1}{(1+\rho)^{T-k}}$$

$$\times \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln(W_k(\alpha_k R_k^{\xi_k} + (1-\alpha_k)R)) + M_{\xi_{k+1}}^{(T-k-1)} \mid \xi_k = i, W_k = w]$$

$$= \frac{1}{(1+\rho)^{T-k}}$$

$$\times \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln W_k + \ln(\alpha_k R_k^{\xi_k} + (1-\alpha_k)R) + M_{\xi_{k+1}}^{(T-k-1)} \mid \xi_k = i, W_k = w]$$

$$= \frac{1}{(1+\rho)^{T-k}}\{\ln w + \max_{0 \le \alpha_{T-1} \le 1} \mathbb{E}[\ln(\alpha_k R_k^i + (1-\alpha_k)R)] + \mathbb{E}_k M_{\xi_{k+1}}^{(T-k-1)}\}$$

$$= \frac{1}{(1+\rho)^{T-k}}\left[\ln w + M_i^{(1)} + \sum_{j=1}^{s} p_{ij} M_j^{(T-k-1)}\right]$$

$$= \frac{1}{(1+\rho)^{T-k}}(\ln w + M_i^{(T-k)}),$$

where $\mathbb{E}_k$ represents expectation under the condition given the information up to time k. Therefore, the conclusion is also true for $n = k$. By induction, the theorem holds.

4. Properties of the Solutions

In Section 3, we obtain expression forms of the value functions and the optimal investment strategy. Our objective value function is maximum expected discounted utility, which is given by

$$V_0(i, w) = \frac{1}{(1+\rho)^T}\left[\ln w + M_i^{(T)}\right].$$

We can see that the solution to our utility maximization problem is intimately related to the functions $M_i^{(\cdot)}$ and its maximizer α_i^* $(i \in Z)$. Therefore, it is necessary for us to study their properties carefully in this section.

Definition 4.1. Let X and Y be two random variables such that

$$\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)]$$

for all increasing and concave functions $f : \mathbb{E} \to \mathbb{R}$, provided the expectations exist. Then X is said to be *smaller than* Y in the second order stochastic order, and it is denoted as $X \le_{SSD} Y$. If distribution functions of X and Y are denoted by $G(\cdot)$ and $F(\cdot)$, then $X \le_{SSD} Y$ can also be denoted as $G \le_{SSD} F$.

Remark 4.1. For more survey on stochastic orders and their relationships, we could refer to Shaked and Shanthikumar [15].

The following proposition builds a relationship between $M_i^{(1)}$, $M_j^{(1)}$, $i, j \in Z$. It lays nice foundation for later part of this paper.

Proposition 4.1. *Let* $i, j \in Z$. *If* $F_j \leq_{SSD} F_i$, *then*

$$M_j^{(1)} \leq M_i^{(1)}.$$

Proof. Suppose $F_j \leq_{SSD} F_i$. First, we note that

$$\begin{aligned}\tilde{M}_j(0) &= \mathbb{E}[\ln R] \\ &= \tilde{M}_i(0),\end{aligned} \tag{4.1}$$

and for any fixed $\alpha : 0 < \alpha \leq 1$, the function $f(x) = \ln(\alpha x + (1-\alpha)R)$ is increasing and concave. By the definition of second order stochastic order, we get

$$\begin{aligned}\tilde{M}_j(\alpha) &= \mathbb{E}[\ln(\alpha R^j + (1-\alpha)R)] \\ &= \mathbb{E}[f(R^j)] \\ &\leq \mathbb{E}[f(R^i)] \\ &= \mathbb{E}[\ln(\alpha R^i + (1-\alpha)R)] \\ &= \tilde{M}_i(\alpha).\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we conclude that $\tilde{M}_j(\alpha) \leq \tilde{M}_i(\alpha)$ for all $\alpha \in [0, 1]$. Thus, we have

$$\begin{aligned}M_j^{(1)} &= \tilde{M}_j(\alpha_j^*) \\ &= \max_{0 \leq \alpha \leq 1} \tilde{M}_j(\alpha) \\ &\leq \max_{0 \leq \alpha \leq 1} \tilde{M}_i(\alpha) \\ &= \tilde{M}_i(\alpha_i^*) \\ &= M_i^{(1)},\end{aligned}$$

which means the conclusion holds.

Note that, in general, the condition $F_j \leq_{SSD} F_i$ is not sufficient to guarantee that $M_j^{(n)} \leq M_i^{(n)}$ for $n > 1$. In the subsequent definition, the structure of the transition matrix P also plays a significant role.

Definition 4.2. Let $\mathbf{A} = (a_{ij})$ be an $m \times m$ matrix, supposing that its elements satisfy $a_{ij} \geq 0$ and $\sum_{j=1}^{m} a_{ij} = 1$. Then $\mathbf{A}$ is said to be *stochastically monotone* if

$$\sum_{k=r}^{m} a_{jk} \leq \sum_{k=r}^{m} a_{ik}$$

for all $1 \leq i,\ j,\ r \leq m$ with $j \leq i$.

The following lemma provides a property of a stochastically monotone matrix, which will be used later.

Lemma 4.1. *Let* $\mathbf{A} = (a_{ij})$ *be an* $m \times m$ *matrix*, $a_{ij} \geq 0$ *and* $\sum_{j=1}^{m} a_{ij} = 1$, *and assume that* $\mathbf{A}$ *is stochastically monotone*. *Then*, *for any non-negative increasing* (*decreasing resp.*) *column vector* $\mathbf{c} = (c_1, c_2, \ldots, c_m)$, *the column vector* $\mathbf{Ac}$ *is also increasing* (*decreasing resp.*).

Proof. Suppose that $\mathbf{c}$ is increasing. Put $\mathbf{B} = (B_1, B_2, \ldots, B_m) = \mathbf{Ac}$, it is sufficient to prove $B_1 \leq B_2$. Obviously,

$$B_1 = \sum_{j=1}^{m} a_{1j} c_j,$$

$$B_2 = \sum_{j=1}^{m} a_{2j} c_j,$$

$\forall r : 1 \leq r \leq m$, put $D_r = \sum_{j=r}^{m} (a_{2j} - a_{1j}) c_j$. Hence, $B_1 \leq B_2$ is equivalent to $D_1 \geq 0$. We will prove $D_r \geq \sum_{j=r}^{m} (a_{2j} - a_{1j}) c_r$ by induction. When $r = m$, due to the fact that $\mathbf{A}$ is stochastically monotone, we have

$$\begin{aligned} D_m &= (a_{2m} - a_{1m})c_m \\ &\geq 0. \end{aligned}$$

We assume that when $r = k$ $(2 \leq k \leq m)$,

$$\begin{aligned} D_k &\geq \sum_{j=k}^{m} (a_{2j} - a_{1j})c_k \\ &\geq 0, \end{aligned}$$

where the second inequality follows from that $\mathbf{A}$ is stochastically monotone. Furthermore, when $r = k - 1$,

$$\begin{aligned} D_{k-1} &= (a_{2,k-1} - a_{1,k-1})c_{k-1} + D_k \\ &\geq (a_{2,k-1} - a_{1,k-1})c_{k-1} + \sum_{j=k}^{m} (a_{2j} - a_{1j})c_k \\ &\geq (a_{2,k-1} - a_{1,k-1})c_{k-1} + \sum_{j=k}^{m} (a_{2j} - a_{1j})c_{k-1} \\ &= \sum_{j=k-1}^{m} (a_{2j} - a_{1j})c_{k-1} \\ &\geq 0, \end{aligned}$$

where the second and the last inequality due to the same reason as mentioned above. Therefore, $D_1 \geq 0$ is true by induction, which means that $\mathbf{Ac}$ is also increasing.

The next proposition gives a sufficient condition to allow us to rank $M_1^{(n)}, M_2^{(n)}, \ldots, M_s^{(n)}$ when $n > 1$.

Proposition 4.2. *Suppose that for any* $i, j \in Z$ *with* $i \neq j$, *we have either* $F_j \leq_{SSD} F_i$ *or* $F_i \leq_{SSD} F_j$, *i.e.,* $\leq_{SSD}$ *is a total order in all random variables* $\{R^i, i \in Z\}$. *By the transitivity of the total order, we may assume without loss of* *generality that*

$$F_1 \le_{SSD} F_2 \le_{SSD} F_3 \le_{SSD} \dots \le_{SSD} F_s. \tag{4.3}$$

If the transition matrix P is stochastically monotone, then whenever $n > 1$ and $i, j \in Z$ with $i > j$, we have

$$M_j^{(n)} \le M_i^{(n)}.$$

Proof. We prove the conclusion by induction. Following from Proposition 4.1 and (4.3), we have

$$M_1^{(1)} \le M_2^{(1)} \le \cdots \le M_s^{(1)}.$$

Suppose that $M_1^{(k)} \le M_2^{(k)} \le \cdots \le M_s^{(k)}$ holds for some $k = 1, 2, \dots, T-1.$ If P is stochastically monotone, then for any $j \le i,$ following from Lemma 4.1, we have

$$\begin{aligned} M_j^{(k+1)} &= M_j^{(1)} + \sum_{l=1}^{s} p_{jl} M_l^{(k)} \\ &\le M_j^{(1)} + \sum_{l=1}^{s} p_{il} M_l^{(k)} \\ &\le M_i^{(1)} + \sum_{l=1}^{s} p_{il} M_l^{(k)} \\ &= M_i^{(k+1)}, \end{aligned}$$

which means that $M_1^{(k+1)} \le M_2^{(k+1)} \le \cdots \le M_s^{(k+1)}$. By induction, we finish whole proof of the conclusion.

The intuitive meaning of the condition $F_j \le_{SSD} F_i$ is that the investment environment in regime i is better than that in regime j, which means that the random return in regime i is more favorable than that in regime j. It is natural to ask whether we should invest a larger proposition of our wealth into the risky asset. In order to study this question, we first give a lemma below.

Lemma 4.2. *Let* $i \in Z$.

(1) *The function* $\tilde{M}_i(\alpha)$ *is differential with respect to* α *on the open interval* $(0, 1)$;

(2) *If the function* $\tilde{M}_i(\alpha)$ *achieves its maximum at* $\alpha_i^* \in (0, 1)$, *then*

$$R\mathbb{E}[(\alpha_i^* R^i + (1 - \alpha_i^*)R)^{-1}] = 1.$$

Proof. (1) For any $0 < \varepsilon_1 < \varepsilon_2 < 1$. Recall that $\tilde{M}_i(\alpha) = \mathbb{E}[\ln(\alpha R^i + (1 - \alpha)R)]$. The derivative of the expression inside the expectation with respect to α is given by

$$\frac{R^i - R}{\alpha R^i + (1 - \alpha)R}.$$

For $\alpha : \varepsilon_1 < \alpha < \varepsilon_2$, noting that

$$\left|\frac{R^i - R}{\alpha R^i + (1 - \alpha)R}\right| \le \frac{|R^i - R|}{(1 - \alpha)R}$$

$$\le \frac{R^i + R}{(1 - \varepsilon_2)R},$$

where the last fraction is integrable by our assumption that R^i is integrable. This implies that the collection of random variables

$$\{(\alpha R^i + (1 - \alpha)R)^{-1}(R^i - R)\}_{\varepsilon_1 < \alpha < \varepsilon_2}$$

is uniformly integrable, hence, $\tilde{M}_i(\alpha)$ is differential on the open interval $(\varepsilon_1, \varepsilon_2)$. Since ε_1 and ε_2 are arbitrarily chosen on $(0, 1)$, we conclude that $\tilde{M}_i(\alpha)$ is differential on $(0, 1)$.

(2) As $\tilde{M}_i(\alpha)$ is differential on $(0, 1)$, for $h : 0 < h < 1$ we may denote the derivative $\frac{d}{d\alpha}\tilde{M}_i(\alpha)|_{\alpha=h}$ as $D_i(h)$, where $D_i(h) = \mathbb{E}[(\alpha R^i + (1 - \alpha)R)^{-1}(R^i - R)]$. If the maximizer α_i^* lies in the open interval $(0, 1)$, then we have the first order

necessary condition $D_i(\alpha_i^*) = 0,$ which means

$$\mathbb{E}[(\alpha_i^* R^i + (1-\alpha_i^*)R)^{-1}(R^i - R)] = 0.$$

By multiplying both sides by $\alpha_i^*,$ then adding the term $R\mathbb{E}[(\alpha_i^* R^i + (1-\alpha_i^*)R)^{-1}]$ to both sides, we can obtain the conclusion.

Now we return to the question put forward above Lemma 4.2, the following proposition gives an affirmative answer to it.

Proposition 4.3. *Let* $i,\ j \in Z.$ *If* $F_j \leq_{SSD} F_i,$ *then*

$$\alpha_j^* \leq \alpha_i^*.$$

Proof. In order to prove $\alpha_j^* \leq \alpha_i^*,$ we need to classify the discussions into the following three cases: (i) $\alpha_j^* = 0,$ (ii) $0 < \alpha_j^* < 1,$ (iii) $\alpha_j^* = 1.$

In fact, for case (i), the result holds obviously.

For case (ii), due to the concavity of $\tilde{M}_i(\cdot).$ Hence, it is sufficient to prove $D_i(\alpha_j^*) \geq D_i(\alpha_i^*) = 0,$ which means that it is enough to prove that $D_i(\alpha_j^*)$ is non-negative. Indeed, following from Lemma 4.2, we have

$$\begin{aligned}
D_i(\alpha_j^*) &= \mathbb{E}[(\alpha_j^* R^i + (1-\alpha_j^*)R)^{-1}(R^i - R)] \\
&= \frac{1}{\alpha_j^*}\mathbb{E}[(\alpha_j^* R^i + (1-\alpha_j^*)R)^{-1}(\alpha_j^* R^i + (1-\alpha_j^*)R - R)] \\
&= \frac{1}{\alpha_j^*}\{1 - R\mathbb{E}[(\alpha_j^* R^i + (1-\alpha_j^*)R)^{-1}]\} \\
&= \frac{R}{\alpha_j^*}\{\mathbb{E}[(\alpha_j^* R^j + (1-\alpha_j^*)R)^{-1}] - \mathbb{E}[(\alpha_j^* R^i + (1-\alpha_j^*)R)^{-1}]\} \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the fact that $F_j \leq_{SSD} F_i$ and

$$f(x) = -(\alpha_j^* x + (1 - \alpha_j^*) R)^{-1}$$

is increasing and concave on $[0, +\infty)$.

For case (iii), due to the concavity of the function $\tilde{M}_i(\cdot)$ on the interval $[0, 1]$, we have the following relationship:

$$D_i(1) \geq 0 \Leftrightarrow \alpha_i^* = 1, \quad i \in Z. \tag{4.4}$$

Thus, $D_j(1)$ is non-negative if $\alpha_j^* = 1$. Define function $h(x) = 1 - Rx^{-1}$ on $(0, +\infty)$. It is an increasing and concave function, and $\mathbb{E}[h(R^i)] = D_i(1)$. Then the condition $F_j \leq_{SSD} F_i$ implies

$$\begin{aligned} D_i(1) &= \mathbb{E}[h(R^i)] \\ &\geq \mathbb{E}[h(R^j)] \\ &= D_j(1) \\ &\geq 0. \end{aligned}$$

Thus, $D_i(1)$ is non-negative. Using (4.4) again, we have $\alpha_i^* = 1$.

References

[1] P. P. Boyle and H. L. Yang, Asset allocation with time variation in expected returns, Insurance: Math. Econom. 21 (1997), 201-218.

[2] K. C. Cheung and H. L. Yang, Asset allocation with regime-switching: discrete-time case, Astin Bull. 34(1) (2004), 99-111.

[3] D. Duffie and R. Kan, A yield-factor model of interest rates, Math. Fin. 6 (1996), 379-406.

[4] R. J. Elliott and J. Hinz, Portfolio analysis, hidden Markov models and chart analysis by PF-diagrams, Internat. J. Theo. Appl. Fin. 5 (2002), 385-399.

[5] R. J. Elliott and P. E. Kopp, Mathematics of Financial Markets, 2nd Ed., Springer, 2004.

[6] R. J. Elliott and J. van der Hoek, An application of hidden Markov models to asset allocation problems, Finance Stochas. 3 (1997), 229-238.

[7] R. J. Elliott, L. L. Chan and T. K. Siu, Option pricing and Esscher transform under regime switching, Ann. Finan. 1(4) (2005), 423-432.

[8] R. J. Elliott, W. C. Hunter and B. M. Jamieson, Financial signal processing, Internat. J. Theo. Appl. Finan. 4 (2001), 567-584.

[9] X. Guo, Information and option pricings, Quantitative Finan. 1 (2001), 38-44.

[10] J. D. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, Econometri. 57(2) (1989), 357-384.

[11] H. Markowitz, Portfolio selection, J. Finan. 7 (1952), 77-91.

[12] R. C. Merton, Lifetime portfolio selection under uncertainty: the continuous time case, Review Econom. Statist. 51(3) (1969), 247-257.

[13] R. C. Merton, Optimal consumption and portfolio rules in a continuous-time model, J. Econo. Theo. 3 (1971), 373-413.

[14] S. R. Pliska, Introduction to Mathematical Finance, United States, Blackwell Publishing, 1997.

[15] M. Shaked and J. G. Shanthikumar, Stochastic Orders, Springer, 2007.

[16] L. R. Sotomayor and A. Cadenillas, Explicit solutions of consumption investment problems in financial markets with regime switching, Math. Fin. 19(2) (2009), 251-279.

[17] G. Yin and X. Y. Zhou, Markowitz mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits, IEEE Trans. Automatic Contr. 49 (2003), 349-360.

[18] X. Y. Zhou and G. Yin, Markowitz mean-variance portfolio selection with regime switching: a continuous-time model, SIAM J. Contr. Optim. 42 (2004), 1466-1482.