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COMPACT HARMONIC FINSLER MANIFOLDS WITH FINITE FUNDAMENTAL GROUPS

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Abstract

We prove that compact harmonic reversible Finsler manifolds with finite fundamental groups must be Riemannian.

0. Introduction

In recent years, Finsler geometry has developed rapidly in its global and analytic aspects. The present main work is to generalize and improve some famous theorems of Riemann geometry to the Finsler setting.

A Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius. In the Riemannian cases, Szabò [9] proved that a compact simply connected harmonic Riemannian manifold is isometric to one of the compact rank-one symmetric spaces. By using the volume comparison, we extend this result to Finsler manifolds in this article.

Theorem. *If* (M, F) *is a compact harmonic reversible Finsler manifold with finite fundamental group, then* F *is a Riemannian metric. In fact, the universal covering space of* M *is isometric to one of the compact rank-one symmetric Riemannian spaces.*

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A Finsler manifold is *locally symmetric* if the geodesic reflection is a locally isometry of the Finsler metric. It is obvious that the geodesic reflection induces the minus identity on the tangent spaces, therefore, complete locally symmetric Finsler manifolds have harmonic reversible Finsler metrics. Thus the following result is a straightforward consequence of the main theorem (cf. [4]).

Corollary. *A compact simply connected locally symmetric Finsler manifold is Riemannian.*

We do not know whether our results extend to non-reversible Finsler metrics as several arguments only work in the reversible case. It would be interesting to clarify this point.

1. Preliminaries

In this section, we recall some basics in Finsler geometry and prove some auxiliary facts. We follow the presentation in [5], where most concepts are developed from Riemannian point of view. We refer to [8] as more exhaustive references in Finsler geometry.

A function $F : TM \to [0, \infty)$ will be called a *Finsler metric* on the manifold M if it is smooth outside the zero section and its restriction to each tangent space T_xM is a quadratically convex norm $F(x, \cdot)$. Finsler metrics for which all norms $F(x, \cdot)$ are symmetric will be called *reversible* Finsler metrics.

By a *geodesic* we always mean an affine geodesic, i.e., a constant-speed one. For a nonzero $v \in TM$, we denote by γ_v the unique geodesic with initial velocity $\dot{\gamma}_v(0) = v$. For every $p \in M$ and $v \in T_pM \setminus \{0\}$ there is a unique positive definite quadratic form g_v on T_pM such that g_v and $F^2|_{T_pM}$ agree to second order at v. If V is a non-vanishing vector field on an open set $U \subset M$, then the family $\{g_{V(p)}\}_{p \in U}$ of quadratic forms defines a Riemannian metric g_V on U. If γ is an embedded geodesic and V extends the velocity field of γ to a neighborhood of γ, then we call g_V an *osculating Riemannian metric* for γ and denote it by g^{γ}. Note that g^{γ} is uniquely determined at every point on γ but the extension to a neighborhood is not unique.

By [2, Sect. 5.5], there is only one reasonable notion of the volume for Riemannian manifolds. However, the situation is different in Finsler geometry. The Finsler volume can be defined in various ways and essentially different results may be obtained, e.g., [2, 8]. Therefore, it is an interesting and important problem to investigate the relations between the volumes and the geometric properties on a Finlser manifold.

The Busemann-Hausdorff volume vol^{bh} of a Finsler space is that multiple of the Lebesgue measure for which the volume of the unit ball equals the volume of Euclidean unit ball. Using Brunn-Minkowski theory, Busemann proved that the Busemann-Hausdorff volume of a Finsler space equals its Hausdorff measure. Hence, from the viewpoint of metric geometry, this is a very natural definition.

Another volume that is used frequently in Finsler geometry is the so-called Holmes-Thompson volume. The Holmes-Thompson volume vol^{ht} of a compact Finsler space is the symplectic volume of the unit co-disc bundle divided by the volume of the Euclidean unit ball. In the case of Riemannian metrics, all unit tangent spaces are isometric to the Euclidean spheres, and we have

$$\mathrm{vol}^{ht}(M,\, F) = \mathrm{vol}^{bh}(M,\, F).$$

On the other hand, in a general Finsler metric, unit tangent spaces may not be isometric to each other, and hence one can not expect the equality. We instead have the following theorem.

Theorem 1.1 [3]**.** *Let* $(M,\, F)$ *be a compact reversible Finsler manifold. Then*

$$\mathrm{vol}^{ht}(M,\, F) \le \mathrm{vol}^{bh}(M,\, F),$$

with equality if and only if F is a Riemannian metric.

There exist counterexamples to the inequality when F is nonreversible, e.g., [6].

A Finsler manifold is called a $C_{2\pi}$- *manifold*, if all geodesics are closed and of the same length 2π. The following statements are standard whose proofs can be found also in [5].

Theorem 1.2. *If* (M, F) *be an* n*- dimensional Finsler* $C_{2\pi}$*- manifold, then the ratio*

$$i(M) = \frac{\mathrm{vol}^{ht}(M, F)}{\mathrm{vol}^{ht}(\mathbb{S}^n, g_0)}$$

is an integer.

Here $(\mathbb{S}^n, g_0)$ is the canonical Riemannian sphere $\mathbb{S}^n$ of radius one in $\mathbb{R}^{n+1}$.

Remark 1.3. Under the assumption of Theorem 1.2, if M is homeomorphic to one of the compact rank-one symmetric spaces $(\mathbb{P}, g_0)$, i.e., $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{C}aP^2$, Weinstein, Yang, and Reznikov showed that

$$\mathrm{vol}^{ht}(M, F) = \mathrm{vol}^{ht}(\mathbb{P}, g_0).$$

2. Harmonic Finsler Manifolds

A compact Finsler manifold is called a *Blaschke manifold*, if every minimal geodesic of length less than the diameter is the unique shortest path between any of its points. Equivalently, for which all cut loci are round spheres of constant radius and dimension. Therefore the Blaschke condition implies the $C_{2\pi}$- condition, up to a scaling of the metric.

Remark 2.1. For a Blaschke manifold the exponential map restricted to the unit tangent sphere defines a great sphere foliation. Since every great sphere foliation of sphere is homeomorphic to a Hopf fibration, a simply connected Blaschke manifold is actually homeomorphic to compact rank-one symmetric spaces [7, 10].

For a nonzero $v \in TM$ the mean curvature $m_t(v)$ of geodesic sphere $S(\gamma_v(0), t)$ of radius t about geodesic $\gamma_v(t)$ has following Taylor expansion

$$m_t(v) = \frac{n-1}{t} - S(v) - \frac{1}{3}(\mathrm{Ric}(v) + 3S(v))t + O(t),$$

where S is S-curvature. Let $\hat{m}_t(v)$ be denote the mean curvature of geodesic sphere $S(\gamma_v(0), t)$ in g^{γ_v} with respect to normal vector $\dot{\gamma}_v(t)$. Then

$$m_t(v) = \hat{m}_t(v) - S(\dot{\gamma}_v(t))$$

$$= \frac{d}{dt}[\ln \eta_t(v)],$$

where $\eta_t(v)$ is the Busemann-Hausdorff volume density of geodesic sphere $S(\gamma_v(0), t)$ around $\gamma_v(t)$. From these identities, we can estimate $m_t(v)$ under a Ricci curvature bound and an S-curvature bound. Then we can establish a volume comparison on the metric balls [8].

A Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius. Hence the harmonic Finsler manifolds have Einstein metrics and zero S-curvature. A historical break in the theory of harmonic Riemannian manifolds was made by Allamigeon when he proved the following: A simply connected harmonic Riemannian manifold is either diffeomorphic to Euclidean space or is a Blaschke Finsler manifold. The following theorem is to put them in a Finsler-geometric setting. For the sake of completeness we sketch the proof.

Theorem 2.2. *A simply connected reversible harmonic Finsler manifold M is either diffeomorphic to Euclidean space or is a Blaschke Finsler manifold.*

Proof. Suppose there is no conjugate points. Then exponential map is a covering map and since M is simply connected, a diffeomorphism. So take a $0 \neq v_0 \in T_pM$ and an $r_0 \in \mathbb{R}$ such that the first conjugate point along γ_{v_0} is $\gamma_{v_0}(r_0)$. Then the first conjugate point along γ_v is $\gamma_v(r_0)$ for all $v \in T_pM$, since the mean curvature is radial. Note that r_0 is the same for every point in M. This means that M is a Blaschke manifold by the Allamigeon-Warner theorem, cf. [1, Corollary 5.31]. □

Now we are ready to prove main theorem using Theorems 1.1, 2.2 and Remarks 1.3, 2.1.

Theorem 2.3. *If* (M, F) *is a compact harmonic reversible Finsler manifold with finite fundamental group*, *then F is a Riemannian metric*.

Proof. Let $\tilde{M}$ be the universal covering space of M. By Theorem 2.2, we know $\tilde{M}$ is a Blaschke $C_{2\pi}$- manifold, up to a scaling of the metric, and by Remark 2.1 we have that $\tilde{M}$ is homeomorphic to one of the compact rank one symmetric spaces $\mathbb{P}$. Then applying Remark 1.3 gives

$$\mathrm{vol}^{ht}(\mathbb{P},\ g_0) = \mathrm{vol}^{ht}(\tilde{M},\ F).$$

Since $\tilde{M}$ is a harmonic Finsler manifold, for all nonzero $v \in T\tilde{M}$, $t \in \mathbb{R}^+$, we obtain

$$\begin{aligned} S(\dot{\gamma}_v(t)) &= \hat{m}_t(v) - m_t(v) \\ &= 0, \end{aligned}$$

and the osculating Riemannian space $(\tilde{M} \setminus \{\gamma_v(0)\},\ g^{\gamma_v})$ is a harmonic Riemannian manifold. On the other hand, in the case of rank one symmetric Riemannian manifold, we have

$$\mathbb{S}^n : \eta_t = \sin^{n-1} t;$$

$$\mathbb{C}P^n : \eta_t = \sin t(1 - \cos t)^{\frac{n-2}{2}};$$

$$\mathbb{H}P^n : \eta_t = \sin^3 t(1 - \cos t)^{\frac{n-4}{2}};$$

$$\mathbb{C}aP^2 : \eta_t = \sin^7 t(1 - \cos t)^4.$$

Szabò [9] remarked that these are *only possibilities* for a compact harmonic Riemannian manifold. Since

$$\frac{d}{dt}[\ln \eta_t(v)] = m_t(v)$$

$$= \hat{m}_t(v)$$

$$= \frac{d}{dt}[\ln \hat{\eta}_t(v)],$$

we have $\eta_t(v) = \hat{\eta}_t(v)$. By the co-area formula, we obtain

$$\mathrm{vol}^{bh}(\tilde{M}, F) = \mathrm{vol}^{bh}(\tilde{M} \setminus \{\gamma_v(0)\}, g^{\gamma_v})$$

$$= \mathrm{vol}^{bh}(\mathbb{P}, g_0).$$

Thus we conclude

$$\mathrm{vol}^{bh}(\mathbb{P}, g_0) = \mathrm{vol}^{ht}(\mathbb{P}, g_0)$$

$$= \mathrm{vol}^{ht}(\tilde{M}, F)$$

$$\leq \mathrm{vol}^{bh}(\tilde{M}, F)$$

$$= \mathrm{vol}^{bh}(\mathbb{P}, g_0).$$

We note that the third line is obtained from Theorem 1.1, and hence we obtain

$$\mathrm{vol}^{ht}(\tilde{M}, F) = \mathrm{vol}^{bh}(\tilde{M}, F).$$

Then by the equality case of Theorem 1.1, F is a Riemannian metric. □

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