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JENSEN AND HERMITE-HADAMARD INCLUSIONS FOR STRONGLY CONCAVE SET-VALUED MAPS

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Abstract

Counterparts of the classical integral and discrete Jensen inequalities and the Hermite-Hadamard Theorem and its converse, for strongly concave set-valued maps, are presented.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let c be a positive number. A function $f : I \to \mathbb{R}$ is called *strongly convex* with modulus c if

$$f(tz + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \quad (1)$$

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for all $x, y \in I$ and $t \in (0, 1)$; f is called *strongly concave* with modulus c if $-f$ is strongly convex with modulus c. Strongly convex functions have been introduced by Polyak [16] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in [9], [13], [17] and [23].

Recently, Huang [6] extended the definition (1) of strongly convex functions to set-valued maps. Some properties and applications of them, can be found in [4], [7] and [15].

Roughly speaking, in the case of set-valued maps convexity and concavity are different to the case of real-valued functions, in view of the fact that if F is strongly convex set-valued map, then $-F$ is also strongly convex and although some properties of strongly convex and strongly concave set-valued maps are similar, they hold, in general, under different assumptions and have to be proved separately. Recently, strongly concave set-valued maps were introduced in [11] and the authors exhibited some properties of them. Our paper is related to [15] where analogous results for strongly convex set-valued maps are presented.

2. Preliminaries

Throughout this paper Y is a Banach space, B is the closed unit ball in Y, $I \subset \mathbb{R}$ denotes an interval and c is a positive constant.

Denote by $n(Y)$ the family of all nonempty subset of Y, and by $conv(Y)$ and $cconv(Y)$ the subfamilies of $n(Y)$ of all convex and compact convex sets, respectively.

A set-valued map $F : I \to n(Y)$ is called *strongly concave* with modulus c, if

$$F(tx + (1-t)y) + cy(1-t)(x-y)^2 B \subset tF(x) + (1-t)F(y), \tag{2}$$

for all $x, y \in I$ and $t \in (0, 1)$ (see [11]).

F is called *concave* if it satisfies (2) with $c = 0$ (see, for instance [12], [14], [18]).

A function $f : I \to Y$ is called a *selection* of $F : I \to n(Y)$ if $f(x) \in F(x)$ for every $x \in I$.

The following lemma characterizes strongly concave set-valued maps with values in $cconv(\mathbb{R})$ and shows connections between conditions (1) and (2) (see [15] where an analogous result for strongly convex set-valued maps is given).

Lemma 2.1. *If a set-valued map* $F : I \to cconv(\mathbb{R})$ *is strongly concave with modulus* c, *then it has the following form* $F(x) = [f_1(x), f_2(x)]$, *where* $f_1, f_2 : I \to \mathbb{R}$, $f_1 \leq f_2$ *on* I, f_1 *is strongly concave with modulus c and* f_2 *is strongly convex with modulus* c.

Conversely, if $f_1, f_2 : I \to \mathbb{R}$, *where* f_1 *is strongly concave with modulus c and* f_2 *is strongly convex with modulus* c, $f_1 \leq f_2$ *on* I, *then, the set-valued maps* F_1, F_2, F_3 *and* F_4 *defined by* $F_1(x) = [f_1(x), f_2(x)]$, $F_2(x) = [f_1(x), +\infty)$, $F_3(x) = (-\infty, f_2(x)]$, $F_4(x) = (-\infty, +\infty)$, $x \in I$, *are strongly concave with modulus c.*

Proof. Assume first that $F : I \to cconv(\mathbb{R})$ is strongly concave with modulus c. Then, for every $x \in I$, $F(x)$ is a closed and bounded interval of $\mathbb{R}$ thus, $F(x) = [f_1(x), f_2(x)]$, $\forall x \in I$.

In order to prove that $f_1(x) = \inf F(x)$ is strongly concave with modulus c, notice that by (2), we have

$$F(tx + (1-y)y) + ct(1-t)(x-y)^2[-1, 1] \subset tF(x) + (1-t)F(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

Thus

$$\inf(tF(x) + (1-t)F(y)) \leq \inf(F(tx + (1-t)y) + ct(1-t)(x-y)^2[-1, 1])$$

$$\leq \inf F(tx + (1-t)y) - ct(1-t)(x-y)^2$$

and, consequently

$$tf_1(x) + (1-t)\, f_1(y) + ct(1-t)(x-y)^2 \le f_1(tx + (1-t)\, y),$$

which shows that f_1 is strongly concave with modulus c. The fact that $f_2(x) = \sup F(x)$ is strongly convex, with modulus c, is proved similarly.

This finishes the first part of proof.

Now, suppose that $f_1,\ f_2 : I \to R,$ where f_1 is strongly concave with modulus c and f_2 is strongly convex with modulus c. We shall show that $F_2(x) = [\, f_1(x), +\infty)$ is strongly concave with modulus c (the proofs in the remaining cases are similar). Indeed,

$$F_2(tx + (1-t)\, y) + ct(1-t)(x-y)^2[-1, 1]$$

$$= [\, f_1(tx + (1-t)\, y) - ct(1-t)(x-y)^2, +\infty]$$

$$\subset [tf_1(x) + (1-t)\, f_2(y), +\infty]$$

$$= tF_2(x) + (1-t)\, F_2(y),$$

which finishes the proof. □

3. The Jensen Inclusion

In [9], the following version of the Jensen inequality was proved: *If* $f : I \to \mathbb{R}$ *is strongly convex with modulus* c, *then*

$$f\left(\int_X \varphi(w)\, d\mu\right)$$

$$\le \int_X f(\varphi(w))\, d\mu - c\int_X (\varphi(w) - m)^2\, d\mu, \tag{3}$$

where $m = \int_X \varphi(w)\, d\mu,$ *for every probability measure space* (X, Σ, μ) *and each*

μ- *integrable function* $\varphi : X \to I$. From this result it readily follows that if f is strongly concave with modulus c, then

$$f\left(\int_X \varphi(w)\,d\mu\right) \geq \int_X f(\varphi(w))\,d\mu + c\int_X (\varphi(w) - m)^2\,d\mu. \tag{4}$$

A counterpart of (3) for set-valued maps was obtained in [15]. In the next theorem, we give a counterpart of (4) for strongly concave set-valued maps. We consider the integral of a set-valued map *in the sense of Aumann*; that is, as the set of all the integrals of all μ- integrable selections of it.

Theorem 3.1. *Let* (X, Σ, μ) *be a probability measure space. If* $F : I \to cconv(Y)$ *is strongly concave with modulus* c, *then for each square-integrable function* $\varphi : X \to I$,

$$f\left(\int_X \varphi(w)\,d\mu\right) + c\int_X (\varphi(w) - m)^2\,d\mu B \subset \int_X F(\varphi(w))\,d\mu, \tag{5}$$

where $m = \int_X \varphi(w)\,d\mu.$

Proof. The proof is divided into two steps.

First, we assume that $Y = \mathbb{R}$, then, by Lemma 2.1, F has the form $F(x) = |\, f_1(x),\ f_2(x)\,|$, $x \in I$, with f_1 is strongly concave with modulus c and f_2 strongly convex with modulus c.

Put $m := \int_X \varphi(w)\,d\mu \in I$ and consider an arbitrary $y \in F(m) = |\, f_1(m),\ f_2(m)\,|$.

Then, using Jensen inequalities (3) and (4) we have

$$\int_X f_1(\varphi(w))\,d\mu + c\int_X (\varphi(w) - m)^2\,d\mu$$

$$\leq f_1\left(\int_X \varphi(w)\,d\mu\right)$$

$$\leq y$$

$$\leq f_2\left(\int_X \varphi(w)\, d\mu\right)$$

$$\leq \int_X f_2(\varphi(w))\, d\mu - c\int_X (\varphi(w) - m)^2\, d\mu,$$

whence

$$\int_X f_1(\varphi(w))\, d\mu \leq y - c\int_X (\varphi(w) - m)^2\, d\mu$$

$$\leq y + c\int_X (\varphi(w) - m)^2\, d\mu$$

$$\leq \int_X f_2(\varphi(w))\, d\mu.$$

Thus

$$y + c\int_X (\varphi(w) - m)^2\, d\mu[-1, 1] \subset \left[\int_X f_1(\varphi(w))\, d\mu,\ \int_X f_2(\varphi(w))\, d\mu\right]$$

$$\subset \int_X F(\varphi(w))\, d\mu$$

and consequently

$$F\left(\int_X \varphi(w)\, d\mu\right) + c\int_X (\varphi(w) - m)^2\, d\mu[-1, 1] \subset \int_X F(\varphi(w))\, d\mu,$$

which finishes the proof in the case $Y = \mathbb{R}$.

Now assume that Y is an arbitrary Banach space. Take a nonzero continuous linear functional $y^* \in Y^*$ and consider the set-valued map $G : I \to cconv(\mathbb{R})$ defined by $G(x) := (\overline{y^* oF})(x)$. This set-valued map is well defined, it is strongly concave with modulus $c\| y^* \|$ and has compact convex values in $\mathbb{R}$. Therefore, by the previous step

$$G\left(\int_X \varphi(w)\,d\mu\right) + c\|\, y^* \,\|\int_X (\varphi(w) - m)^2\, d\mu[-1, 1] \subset \int_X G(\varphi(w))\,d\mu; \qquad (6)$$

that is

$$(\overline{y^* oF})\left(\int_X \varphi(w)\,d\mu\right) + c\|\, y^* \,\|\int_X (\varphi(w) - m)^2\, d\mu[-1, 1] \subset \int_X (\overline{y^* oF})(\varphi(w))\,d\mu. \qquad (6`)$$

Fix a point $b \in B$ and take an arbitrary $y \in \left(\int_X \varphi(w)\,d\mu\right) = F(m)$, then

$$y + c\int_X (\varphi(w) - m)^2\, d\mu b \in F\left(\int_X \varphi(w)\,d\mu\right) + c\int_X (\mu(w) - m)^2\, d\mu B.$$

Hence, by (6`),

$$y^*\left(y + c\int_X (\varphi(w) - m)^2\, d\mu b\right) \in y^*(y) + c\|\, y * \|\int_X (\varphi(w) - m)^2\, d\mu[-1, 1]$$

$$\subset \int_X (\overline{y^* oF})(\varphi(w))\,d\mu$$

$$\subset \overline{y^* \int_X F(\varphi(w))\,d\mu}.$$

Since this condition holds for arbitrary $y^* \in Y^*$ and the set $\overline{y^*\left(\int_X F(\varphi(w))\,d\mu\right)}$ is convex and closed, by the *separation theorem* (see [20, Corollary 2.5.11]) we obtain

$$y + c\int_X (\varphi(w) - m)^2\, d\mu b \in \int_X F(\varphi(w)\,d\mu$$

and since *y* and *b* arbitraries, we conclude that

$$F\left(\int_X \varphi(w)\,d\mu\right) + c\int_X (\varphi(w) - m)^3\, d\mu B \subset \int_X F(\varphi(w))\,d\mu. \qquad \square$$

As a consequence of Theorem 3.1, we obtain the following discrete Jensen inclusion for strongly concave set-valued maps.

Corollary 3.1. *If* $F : I \to cconv(Y)$ *is strongly concave with modulus* c, *then*

$$F\left(\sum_{i=1}^{n} t_i x_i\right) + c\sum_{i=1}^{n} t_i (x_i - m)^2 B \subset \sum_{i=1}^{n} t_i F(x_i)$$

for all $n \in N$, $x_1, x_2, \dots x_n \in I$, $t_1, \dots, t_n > 0$ *with* $\sum_{i=1}^{n} t_i = 1$ *and* $m = \sum_{i=1}^{n} t_i x_i$.

Proof. Suppose that $X = I$, $\varphi(x) = x$, for $x \in I$, and that $x_1, x_2, \dots, x_n \in I$ are distinct points. Moreover, assume that μ is a probability measure concentrated at $x_1, x_2, \dots, x_n$; that is, $\mu(x_i) = t_i > 0$, $i = 1, \dots, n$ and $\sum_{i=1}^{n} t_i = 1$. Then

$$m = \int_X \varphi(w)\, d\mu = \sum_{i=1}^{n} t_i x_i, \quad \int_X (\mu(w) - m)^2 d\mu = \sum_{i=1}^{n} t_i (x_i - m)^2$$

and

$$\int_X F(\varphi(w))\, d\mu = \sum_{i=1}^{n} t_i F(x_i).$$

Now, using the strong concavity of F and substituting the above equalities in (5), we get

$$F\left(\sum_{i=1}^{n} t_i x_i\right) + c\sum_{i=1}^{n} t_i (x_i - m)^2 B \subset \sum_{i=1}^{n} t_i F(x_i),$$

which finishes the proof. □

4. The Hermite-Hadamard Inclusion

It is know that if a function $f : I \to \mathbb{R}$ is strongly convex with modulus c, then it satisfies the version of the Hermite-Hadamard double inequality

$$f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2, \tag{7}$$

for all $a, b, \in I$ with $a < b$ (see [9]). In the case that f is convex then it satisfies (7) with $c = 0.$

It readily follows that if f is strongly concave with modulus $c,$ then the version of the Hermite-Hadamard has the form

$$f\left(\frac{a+b}{2}\right)-\frac{c}{12}(b-a)^2 \geq \frac{1}{b-a}\int_a^b f(x)dx \geq \frac{f(a)+f(b)}{2}+\frac{c}{6}(b-a)^2. \tag{8}$$

Recently, the authors in [15] obtained a counterpart of the above inequality (7) for strongly convex set-valued maps. In this section, we present a counterpart of (8) for strongly concave set-valued maps.

Theorem 4.1. *If a set-valued map* $F : I \to cconv(Y)$ *is strongly concave with modulus* $c,$ *then*

$$F\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2 B \subset \frac{1}{b-a}\int_a^b F(x)dx, \quad a, b, \in I, \quad a < b \tag{9}$$

and

$$\frac{1}{b-a}\int_a^b F(x)dx+\frac{c}{6}(b-a)^2 B \subset \frac{F(a)+F(b)}{2}, \quad a, b \in I, \quad a < b. \tag{10}$$

Proof. To show that condition (9) holds, take $X = [a, b],$ $\varphi : [a, b] \to I$ defined by $\varphi(x) = x$ and $\mu = \frac{1}{b-a}\lambda,$ where λ is the Lebesgue measure on $\mathbb{R}.$ Then

$$\int_X \varphi(x)d\mu = \frac{a+b}{2},$$

$$F\left(\int_X \varphi(x)\,d\mu\right) = F\left(\frac{a+b}{2}\right),$$

$$\int_X (\varphi(x) - m)^2 d\mu = \frac{1}{12}(b-a)^2$$

and

$$\int_X F(\varphi(x))d\mu = \frac{1}{b-a}\int_a^b F(x)\,dx.$$

Using the fact that F is strongly concave with modulus c, by substituting these equalities in (5) we get (9).

To prove (10), consider $a, b \in I$, $a < b$, $f : [a, b] \to Y$ a μ-integrable selection of F and $\rho \in B$.

Express $F(x) = F\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right)$, $x \in [a, b]$, and take

$$f(x) + c\left(\frac{b-x}{b-a}\right)\left(\frac{x-a}{b-a}\right)(b-a)^2\rho$$

$$\in F\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + c\left(\frac{b-x}{b-a}\right)\left(\frac{x-a}{b-a}\right)(b-a)^2 B.$$

By the strong concavity of F, it follows that

$$f(x) + c\left(\frac{b-x}{b-a}\right)\left(\frac{x-a}{b-a}\right)(b-a)^2\rho \in \frac{b-x}{b-a}F(a) + \frac{x-a}{b-a}F(b),$$

hence, there exist $u \in F(a)$ and $v \in F(b)$ such that

$$f(x) + c\left(\frac{b-x}{b-a}\right)\left(\frac{x-a}{b-a}\right)(b-a)^2\rho = \frac{b-x}{b-a}u + \frac{x-a}{b-a}v.$$

On the other hand, by simple calculations we have

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{c}{6}(b-a)^2\rho = \frac{u+v}{2}$$

$$\in \frac{F(a)+F(b)}{2}.$$

Therefore, we conclude that

$$\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{6}(b-a)^2 B \subset \frac{F(a)+F(b)}{2}. \qquad \square$$

5. The Converse of the Hermite-Hadamard Theorem

In [9], the following version for the converse of the Hermite-Hadamard Theorem was proved: if a continuous function $f : I \to \mathbb{R}$ satisfies the left or the right hand-side inequality of (7), then it is strongly convex with modulus c. If f satisfies the same conditions on (7) with $c = 0$, then f is convex.

Similarly, if f is continuous and satisfies the left or the right hand-side inequality of (8), then f is strongly concave with modulus c.

In what follows we assume that Y is a separable Banach space and denote by $bccl(Y)$ the family of all bounded convex closed and non-empty subsets of Y.

Recently, in [15], it was proved a counterpart for the converse of the Hermite-Hadamard Theorem for strongly convex set-valued maps: if $F : I \to bccl(Y)$ is continuous and satisfies

$$\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{12}(a-b)^2 B \subset F\left(\frac{a+b}{2}\right), \quad a, b \in I, \quad a < b$$

or

$$\frac{F(a)+F(B)}{2} + \frac{c}{6}(a-b)^2 B \subset \frac{1}{b-a}\int_a^b F(x)dx, \quad a, b \in I, \quad a < b,$$

then F is strongly convex with modulus c.

In this section, we present the converse of the Hermite-Hadamard Theorem for strongly concave set-valued maps.

Recall that a set-valued map $F : I \to n(Y)$ is said to be *continuous* at a point x_0 if for every neighborhood V of zero in Y there exists a neighborhood U of zero in R such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V$$

for all $x \in (x_0 + U) \cap I$.

Now, we show the converse of the Hermite-Hadamard Theorem for strongly concave set-valued maps.

Theorem 5.1. *If a set-valued map* $F : I \to cconv(Y)$ *is continuous and satisfies*

$$\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{6}(b-a)^2 B \subset \frac{F(a)+F(b)}{2}, \quad a, b \in I, \quad a < b, \tag{11}$$

or

$$F\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 B \subset \frac{1}{b-a}\int_a^b F(x)dx, \quad a, b \in I, \quad a < b, \tag{12}$$

then F is strongly concave with modulus c.

Proof. The idea of the proof is taken from [10, Theorem 8].

Suppose that F is not strongly concave with modulus c, i.e., (2) does not hold. Then, there are $t_0 \in (0, 1)$, $x_1, x_2 \in I$ and $z \in F(t_0x_1 + (1-t_0)x_2) + ct_0(1-t_0)(x_1 - x_2)^2 B$ such that $z \notin t_0F(x_1) + (1-t_0)F(x_2)$.

Since the set $t_0F(x_1) + (1-t_0)F(x_2)$ is convex and closed, by the separation theorem, there exists a continuous linear functional $y^* \in Y^*$ such that

$$y^*(z) > \sup\{y^*(y) : y \in t_0F(x_1) + (1-t_0)F(x_2)\}. \tag{13}$$

Now, if F satisfies (11) (the proof in case that F satisfies (12) is similar), then

$$y^*\left(\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{6}(b-a)^2 B\right) \subset \frac{y^*(F(a)) + y^*(F(b))}{2}. \tag{14}$$

Consider the function $f : I \to \mathbb{R}$ defined by $f(x) := \sup y^*(F(x)),\ \ x \in I.$ Clearly, f is continuous and, in view of (14) and the fact that

$$\int_a^b \sup y^*(F(x))dx = \sup y^* \int_a^b F(x)dx$$

(see [5, Proposition 5.2]), it satisfies

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{c}{6}(b-a)^2 \le \frac{f(a) + f(b)}{2},$$

that is, f is continuous and satisfies the right hand-side inequality of (7). Thus by the converse of Hermite-Hadamard inequalities we have that f is strongly convex with modulus c, in particular,

$$f(t_0x_1 + (1-t_0)x_2) + ct_0(1-t_0)(x_1 - x_2)^2 \le t_0 f(x_1) + (1-t_0)f(x_2).$$

Consequently, by the definition of f, we obtain

$$\begin{aligned} y^*(z) &\le \sup y^*(F(t_0x_1 + (1-t_0)x_2) + ct_0(1-t_0)(x_1 - x_2)^2 B) \\ &\le t_0 \sup y^*(F(x_1)) + (1-t_0)\sup y^*(F(x_2)) \\ &= \sup y^*(t_0F(x_1) + (1-t_0)F(x_2)), \end{aligned}$$

this contradicts (13) and finishes the proof. □

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