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ON THE INTEGER SOLUTIONS OF THE PELL EQUATION $x^2 = 17y^2 - 19^t$

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Abstract

Let $d \neq 1$ be a positive non-square integer and N be any fixed positive integer. Then the equation $x^2 - dy^2 = \pm N$ is known as Pell's equation named after the famous mathematician John Pell. In this paper, we fix d and N to be two Coprimes 17 and 19 and search for non-trivial integer solutions to the equation $x^2 = 17y^2 - 19^t$, $t \in N$ for the different choices of t given by (i) $t = 1$, (ii) $t = 3$, (iii) $t = 5$, (iv) $2k$ and $t = 2k + 5$. Further, recurrence relations on the solutions are obtained.

1. Preliminary Facts

Let $d \neq 1$ be a positive non-square integer and N be any fixed positive integer. Then the equation

$$x^2 - dy^2 = \pm N \tag{A}$$

is known as *Pell's equation* named after the famous Mathematician John Pell (1611-

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1685), who searched for integer solutions to equations of this type in the seventeenth century. Ironically, Pell was not the first to work on this problem, nor did he contribute to our knowledge for solving it. Euler (1707-1783), who brought us the ψ-function, accidentally named the equation after Pell, and the name stuck. For $N = 1,$ the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{B}$$

is known as the *classical Pell equation* and was first studied by Brahma Gupta (598-670) and Bhaskara (1114-1185).

The Pell equation in (B) has infinitely many integer solutions (x_n, y_n) for $n \geq 1.$ The first non-trivial positive integer solution (x_1, y_1) (in this case x_1 or $x_1 + y_1\sqrt{d}$ is minimum) of this equation is called the *fundamental solution*, because all other solutions can be (easily) derived from it. In fact, if (x_1, y_1) is the fundamental solution of $x^2 - dy^2 = 1,$ then the nth positive solution of it, say $(x_n, y_n),$ is defined by the equality

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \tag{C}$$

for integer $n \geq 2.$ (Furthermore, all nontrivial solutions can be obtained considering the four cases $(\pm x_n, \pm y_n)$ for $n \geq 1$).

There are several methods for finding the fundamental solution of Pell's equation $x^2 - dy^2 = 1$ for a positive non-square integer d, e.g., the cyclic method [4, p. 30], known in India in the 12-th century, or the slightly less efficient but more regular English method (17-th century) which produce all solutions of $x^2 - dy^2 = 1$ [4, p. 32]. But the most efficient method for finding the fundamental solution is based on the simple finite continued fraction expansion of $\sqrt{d}$. We can describe it as follows (see [2] and also [6, p. 154]):

Let $[a_0; \overline{a_1, a_2, \ldots, a_r, 2a_0}]$ be the simple continued fraction of $\sqrt{d}$, where $a_0 = \lfloor \sqrt{d} \rfloor.$

Let $p_0 = a_0$, $p_1 = 1 + a_0 a_1$, $q_0 = 1$, $q_1 = a_1$. In general

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2} \tag{D}$$

for $n \geq 2$. Then the fundamental solution of $x^2 - dy^2 = 1$ is

$$(x_1,\ y_1) = \begin{cases} (p_r,\ q_r), & \text{if } r \text{ is odd,} \\ (p_{2r+1},\ q_{2r+1}), & \text{if } r \text{ is even.} \end{cases} \tag{E}$$

To obtain the integer solutions of (A), the initial solution of (A) is incorporated with the general solution (C) of (B) using Brahma Gupta lemma.

On the other hand, in connection with (A) and (B), it is well known that if (u_1, v_1) and $(x_{n-1},\ y_{n-1})$ are integer solutions of $x^2 - dy^2 = \pm N$ and $x^2 - dy^2 = 1$, respectively, then (u_n, v_n) is also a positive solution of $x^2 - dy^2 = \pm N$,

$$u_n + \sqrt{d}\, v_n = (x_{n-1} + \sqrt{d}\, y_{n-1})(u_1 + \sqrt{d}\, v_1) \tag{F}$$

for $n \geq 2$.

Theorem 1 [11]**.** *Let p be a prime. Then the negative Pell's equation*

$$x^2 - py^2 = -1$$

is solvable if and only if $p = 2$ *or* $p \equiv 1 (\mathrm{mod}\ 4)$.

Proof. If the considered equation has a solution $(x,\ y)$, then $p \mid x^2 + 1$. Hence either $p = 2$ or $p \equiv 1 (\mathrm{mod}\ 4)$.

For $p = 2$, $x = y = 1$ is solution. We show that there is solution for each prime $p = 4t + 1$. A natural starting point is the existence of an integral solution $(x_0,\ y_0)$ to the corresponding Pell's equation: $x_0^2 - py_0^2 = 1$. We observe that x_0 odd:

otherwise, $y_0^2 \equiv py_0^2 \equiv 3 \pmod 4$. Thus in the relation

$$x_0^2 - 1 = (x_0 - 1)(x_0 + 1) = py_0^2,$$

factors $x_0 + 1$ and $x_0 - 1$ have greatest common divisor 2, and consequently one of them is a doubled square (to be denoted by $2x^2$) and the other one $2p$ times a square (to be denoted by $2py^2$). The case $x_0 + 1 = 2x^2$, $x_0 - 1 = 2py^2$ is impossible because it leads to a smaller solution of Pell's equation: $x^2 - py^2 = 1$. It follows that $x_0 - 1 = 2x^2$, $x_0 + 1 = 2py^2$, and therefore $x^2 - py^2 = -1$.

2. The Pell Equation $x^2 = 17y^2 - 19^t$

For this particular equation we consider the prime $p = 17$. Since p satisfies all the conditions of Theorem 1, we can conclude that the negative Pell equation $x^2 = 17y^2 - 19^t$ is solvable in integers.

2.1. Choice 1: $t = 1$

The Pell equation is

$$x^2 = 17y^2 - 19 \tag{1}$$

with the initial solution $x_0 = 41$; $y_0 = 10$.

To find the other solutions, consider the more general Pell equation $x^2 = 17y^2 + 1$ whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2} f_n, \quad \tilde{y}_n = \frac{1}{2\sqrt{17}} g_n,$$

where $f_n = (33 + 8\sqrt{17})^{n+1} + (33 - 8\sqrt{17})^{n+1}$; $g_n = (33 + 8\sqrt{17})^{n+1} - (33 - 8\sqrt{17})^{n+1}$, $n = 0, 1, 2, \ldots$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as $x_{n+1} = \frac{1}{2}[41f_n + 10\sqrt{17}g_n]$ and $y_{n+1} = \frac{1}{2\sqrt{17}}[10\sqrt{17}f_n + 41g_n]$. The values of f_n and g_n are found to be

$$f_n = \frac{1}{19}[340y_{n+1} - 82x_{n+1}]; \quad g_n = \frac{1}{19}[20\sqrt{17}x_{n+1} - 82\sqrt{17}y_{n+1}].$$

2.2. Choices 2: $t = 3$

The Pell equation is

$$x^2 = 17y^2 - 6859 \tag{2}$$

with (x_0, y_0) be the initial solution $x_0 = 37$; $y_0 = 22$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{1}{2}[37f_n + 22\sqrt{17}g_n], \quad y_{n+1} = \frac{1}{2\sqrt{17}}[22\sqrt{17}f_n + 37g_n].$$

The values of f_n and g_n are found to be

$$f_n = \frac{1}{6859}[748y_{n+1} - 74x_{n+1}]; \quad g_n = \frac{1}{6859}[44\sqrt{17}x_{n+1} - 74\sqrt{17}y_{n+1}].$$

2.3. Choices 3: $t = 5$

The Pell equation is

$$x^2 = 17y^2 - 2476099 \tag{3}$$

with (x_0, y_0) be the initial solution $x_0 = 6143$; $y_0 = 1538$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained

$$x_{n+1} = \frac{1}{2}[6143f_n + 1538\sqrt{17}g_n], \quad y_{n+1} = \frac{1}{2\sqrt{17}}[1538\sqrt{17}f_n + 6143g_n].$$

The values of f_n and g_n are found to be

$$f_n = \frac{1}{2476099}[52292 y_{n+1} - 12286 x_{n+1}];$$

$$g_n = \frac{1}{2476099}[3076\sqrt{17} x_{n+1} - 12286\sqrt{17} y_{n+1}].$$

2.4. Choices 4: $t = 2k,\ k > 0$

The Pell equation is

$$x^2 = 17y^2 - 19^{2k}, \quad k > 0 \tag{4}$$

with (x_0, y_0) be the initial solution $x_0 = 19^k \cdot 4;\ y_0 = 19^k$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{19^k}{2}[4f_n + \sqrt{17} g_n], \quad y_{n+1} = \frac{19^k}{2\sqrt{17}}[\sqrt{17} f_n + 4g_n].$$

The values of f_n and g_n are found to be

$$f_n = \frac{1}{19^k}[34 y_{n+1} - 8x_{n+1}]; \quad g_n = \frac{1}{19^k}[2\sqrt{17} x_{n+1} - 8\sqrt{17} y_{n+1}].$$

2.5. Choices 5: $t = 2k + 5,\ k > 0$

The Pell equation is

$$x^2 = 17y^2 - 19^{2k+5}, \quad k > 0 \tag{5}$$

with (x_0, y_0) be the initial solution $x_0 = 19^{k-1} \cdot 11827;\ y_0 = 19^{k-1} \cdot 7798$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as

$$x_{n+1} = \frac{19^{k-1}}{2}[11827 f_n + 7798\sqrt{17} g_n],$$

$$y_{n+1} = \frac{19^{k-1}}{2\sqrt{17}}[7798\sqrt{17} f_n + 11827 g_n].$$

The values of f_n and g_n are found to be

$$f_n = \frac{1}{19^{k+6}}[265132y_{n+1} - 23654x_{n+1}];$$

$$g_n = \frac{1}{19^{k+6}}[15596\sqrt{17}x_{n+1} - 23654\sqrt{17}y_{n+1}].$$

The recurrence relations satisfied by the solutions of $x^2 = 17y^2 - 19^t$ are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0; \quad y_{n+2} - 66y_{n+1} + y_n = 0.$$

Conclusion

Solving a Pell's equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell-like equation.

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