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OSCILLATION OF FRACTIONAL DIFFERENCE EQUATIONS WITH DAMPING TERMS

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Abstract

This paper discusses oscillatory behavior of fractional order nonlinear damped difference equations of the form

$$\Delta(a(t)(\Delta^{\alpha}x(t))^{\gamma}) + p(t)(\Delta^{\alpha}x(t))^{\gamma} + q(t)\, f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right) = 0,$$

$$t \in N_{t_0+1-\alpha},$$

Δ^{α} denotes the Riemann-Liouville difference operator of order $0 < \alpha \leq 1$ and $\gamma > 0$ is a quotient of odd positive integers. By means of generalized Riccati

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transformation techniques, we establish some new oscillation criteria for fractional order nonlinear difference equations with damping.

1. Introduction

In recent years, the oscillatory behavior of fractional order differential equations has been the subject of investigation by many authors [2, 6]. The qualitative behavior of solutions of nonlinear difference equations with damping terms is of particular interest [8, 9]. Several recent papers [3-5] dealt with the oscillatory behavior of solutions of fractional difference equations. In this paper, we discuss the oscillatory behavior of fractional order nonlinear damped difference equations of the form

$$\Delta(a(t)(\Delta^{\alpha}x(t))^{\gamma}) + p(t)(\Delta^{\alpha}x(t))^{\gamma} + q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right) = 0,$$

$$t \in N_{t_0+1-\alpha}, \qquad (1)$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order $0 < \alpha \leq 1$ and $\gamma > 0$ is a quotient of odd positive integers. In (1), $a(t)$ is a positive sequence, $p(t)$ is a non negative sequence and $q(t)$ is a non negative sequence possessing a positive subsequence. In this paper, we assume the following conditions.

(H1) $a(t) - p(t) > 0$ for all large t.

(H2) $f : \mathbf{R} \to \mathbf{R}$ such that $xf(x) > 0$ for $x \neq 0$.

(H3) $f : \mathbf{R} \to \mathbf{R}$ such that $f(u) - f(v) = g(u, v)(u - v)$, $u, v \neq 0$ and g is non negative and $g(u, v) \geq \tau > 0$.

(H4) $f : \mathbf{R} \to \mathbf{R}$ such that $xf(x) > 0$ for $x \neq 0$, and non decreasing.

(H5) $f : \mathbf{R} \to \mathbf{R}$ such that $xf(x) > 0$ and $f(x) \geq kx$ for $x \neq 0$, and some $k > 0$.

(H6) $f : \mathbf{R} \to \mathbf{R}$ such that $xf(x) > 0$ and $f(x) \geq kx^{\beta}$ for $x \neq 0$, and some $k > 0$, and β is a quotient of odd positive integers.

In order to discuss the oscillatory properties of (1), we need the following case.

(Ha)

$$\sum_{s=t_0}^{\infty}\left(\frac{1}{a(t)}\prod_{s=t_0}^{t-1}\left(1-\frac{p(t)}{a(t)}\right)\right)=\infty.$$

2. Preliminaries and Basic Lemmas

In this section, we introduce preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 2.1 (see [7])**.** Let $\nu = 0$. Then the ν- th fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s \equiv a \bmod(1)\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu)\bmod(1)$, and $t^{(\nu)} = \frac{\Gamma(t-1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on $N_{a+\nu}$.

Definition 2.2 (see [7])**.** Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ- th fractional Riemann-Liouville difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t)$$

$$= \Delta^{m}\Delta^{-\nu} f(t).$$

In order to discuss our results in Section 3, now we state the following lemma.

Lemma 2.3 (Hardy et al. see [1])**.** *If X and Y are non-negative*, *then*

$$mXY^{m-1} - X^{m} \le (m-1)Y^{m} \quad for \quad m > 1, \tag{2}$$

where equality holds if and only if $X = Y$.

Lemma 2.4 [5]**.** *Let* $x(t)$ *be a solution of* (1) *and let*

$$G(t) = \sum_{s=t_0}^{t-1+a} (t-s-1)^{(-\alpha)} x(s). \tag{3}$$

Then

$$\Delta(G(t)) = \Gamma(1-\alpha)\Delta^{\alpha}(x(t)). \tag{4}$$

3. Main Results

Throughout this section, we will assume that (Ha) holds. To prove our main results, we need the following lemma. We state and prove it here for the sake of completeness.

Lemma 3.1. *Suppose* (H1), (H2) *and* (Ha) *hold. If* $x(t)$ *is a non-oscillatory solution of* (1), *then there is an integer* $t_1 \geq t_0$ *such that* $x(t)\Delta^{\alpha}x(t) > 0$ *for all* $t \geq t_1$.

Proof. Let $x(t)$ be a non-oscillatory solution of (1). Without loss of generality, we may assume that $x(t) > 0$ and $G(t) > 0$ for all large t. We claim that $\Delta^{\alpha}x(t)$ is eventually positive. Indeed, suppose to the contrary that for any large $T \geq t_0$, there is an integer t_1 such that $\Delta^{\alpha}x(t_1) = 0$ or $\Delta^{\alpha}x(t_1) < 0$. In view of (1), we have

$$\begin{aligned}(\Delta^{\alpha}x(t_1))(\Delta(a(t_1)(\Delta^{\alpha}x(t_1))^{\gamma})) &= -p(t_1)(\Delta^{\alpha}x(t_1))^{\gamma+1} - q(t_1)f(G(t_1))\Delta^{\alpha}x(t_1)\\ &\geq -p(t_1)(\Delta^{\alpha}x(t_1))^{\gamma+1}.\end{aligned}$$

Hence

$$(\Delta^{\alpha}x(t_1))((a(t_1+1)(\Delta^{\alpha}x(t_1+1))^{\gamma} - a(t_1)(\Delta^{\alpha}x(t_1))^{\gamma}) \geq -p(t_1)(\Delta^{\alpha}x(t_1))^{\gamma+1}.$$

So

$$(\Delta^{\alpha}x(t_1))(a(t_1+1)(\Delta^{\alpha}x(t_1+1))^{\gamma}) \geq (a(t_1) - p(t_1))(\Delta^{\alpha}x(t_1))^{\gamma+1}.$$

By (H1), $(a(t_1) - p(t_1)) > 0$. When T is sufficiently large, we must have

$\Delta^{\alpha}x(t_1+1) < 0$. By induction, we obtain $\Delta^{\alpha}x(t) < 0$ for all $t \geq t_1$. In the latter case, in view of (1) and (H2), we obtain

$$\begin{aligned}\Delta(a(t_1)(\Delta^{\alpha}x(t_1))^{\gamma}) &= -q(t_1)f(G(t_1))\\ &\leq 0\end{aligned}$$

which implies that $\Delta^{\alpha}x(t_1+1) \leq 0$. If $\Delta^{\alpha}x(t_1+1) < 0$, then based on what we have just observed, we may conclude that $\Delta^{\alpha}x(t) < 0$ for $t \geq t_1+1$. If $\Delta^{\alpha}x(t_1+1) = 0$, then by induction we may conclude that $\Delta^{\alpha}x(t_1+2) \leq 0$. By induction again, we will end up with two situations: either $\Delta^{\alpha}x(t)$ is eventually negative or $\Delta^{\alpha}x(t) = 0$ for $t \geq t_1$. However, the latter case is impossible. Indeed, since $q(t)$ has a positive sub sequence, we may let i be an integer greater that t_1 so that $q(i) > 0$. Then in view of (1), we have

$$\begin{aligned}0 &= \Delta(a(i)(\Delta^{\alpha}x(i))^{\gamma}) + p(i)(\Delta^{\alpha}x(i))^{\gamma} + q(i)f(G(i))\\ &= q(i)f(G(i)) > 0,\end{aligned}$$

which is a contradiction.

If we now define $u(t) = -a(t)(\Delta^{\alpha}x(t))^{\gamma}$ for $t \geq t_2 \geq t_1$ such that $\Delta^{\alpha}x(t_2) < 0$, then from (1), we have

$$\Delta u(t) + \frac{p(t)}{a(t)}u(t) \geq 0$$

thus

$$u(t) \geq u(t_2+1)\prod_{s=t_2+1}^{t-1}\left(1-\frac{p(s)}{a(s)}\right),$$

so

$$\Delta^{\alpha}x(t) \leq -u(t_2+1)^{1/\gamma}\left(\frac{1}{a(t)}\prod_{s=t_2+1}^{t-1}\left(1-\frac{p(s)}{a(s)}\right)\right)^{1/\gamma},$$

$$\Delta^{\alpha}x(t) \leq a(t_2+1)^{1/\gamma}\Delta^{\alpha}x(t_2+1)\left(\frac{1}{a(t)}\prod_{s=t_2+1}^{t-1}\left(1-\frac{p(s)}{a(s)}\right)\right)^{1/\gamma},$$

$$\Delta G(t) \leq \Gamma(1-\alpha)a(t_2+1)^{1/\gamma}\Delta^{\alpha}x(t_2+1)\left(\frac{1}{a(t)}\prod_{s=t_2+1}^{t-1}\left(1-\frac{p(s)}{a(s)}\right)\right)^{1/\gamma}.$$

Summing the last inequality from t_2+1 to $t-1$,

$$G(t)-G(t_2+1)$$

$$\leq \Gamma(1-\alpha)a(t_2+1)^{1/\gamma}\Delta^{\alpha}x(t_2+1)\sum_{s=t_2+1}^{t-1}\left(\frac{1}{a(s)}\prod_{t=t_2+1}^{s-1}\left(1-\frac{p(t)}{a(t)}\right)\right)^{1/\gamma}.$$

Condition (Ha) and $t\to\infty$ implies that $G(t)\to-\infty$, which is a contradiction. The proof is complete.

Theorem 3.2. *Suppose that* $(\mathrm{H1})$, $(\mathrm{H5})$ *and* (Ha) *hold. Furthermore, assume that there exists a positive sequence* $\rho(t)$ *such that for every positive constant* M,

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t}\left(k\rho(s)q(s)-\frac{\Psi^2(s)}{4R(s)}\right)=\infty, \tag{5}$$

where

$$R(t)=\frac{\Gamma(1-\alpha)M^{\frac{\gamma-1}{\gamma}}\rho(t)}{(a(t))^{\frac{1}{\gamma}}}$$

and

$$\Psi(t)=\frac{\alpha(t)\Delta\rho(t)-\rho(t)\,p(t)}{a(t)}.$$

Then every solution of (1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a non-oscillatory solution of (1).

Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1) such that $x(t) > 0$ for large t. In view of Lemma 3.1, then there exists $t_1 \geq t_0$ such that

$$x(t) > 0, \quad \Delta^{\alpha} x(t), \quad \Delta(a(t)(\Delta^{\alpha} x(t))^{\gamma}) \leq 0, \quad t \geq t_1. \tag{6}$$

Define the sequence $w(t)$ by

$$w(t) = \rho(t)\left(\frac{a(t)(\Delta^{\alpha} x(t))^{\gamma}}{G(t)}\right), \quad t \geq t_1. \tag{7}$$

Then $w(t) > 0$ and

$$\Delta w(t) = a(t+1)(\Delta^{\alpha} x(t+1))^{\gamma} \Delta\left[\frac{\rho(t)}{G(t)}\right] + \frac{\rho(t)\Delta(a(t)(\Delta^{\alpha} x(t))^{\gamma})}{G(t)}.$$

Using (1)

$$\Delta w(t) = a(t+1)(\Delta^{\alpha} x(t+1))^{\gamma}\left[\frac{G(t)\Delta\rho(t) - \rho(t)\Delta G(t))}{G(t+1)G(t)}\right] \tag{8}$$

$$- \rho(t)\left[\frac{p(t)(\Delta^{\alpha} x(t))^{\gamma} + q(t) f(G(t))}{G(t)}\right].$$

In view of $(\mathrm{H5})$ and (8), we have

$$\Delta w(t) \leq -k\rho(t)q(t) - \rho(t)\frac{p(t)(\Delta^{\alpha} x(t))^{\gamma}}{G(t)}$$

$$+ \frac{a(t+1)(\Delta^{\alpha} x(t+1))^{\gamma}\Delta\rho(t)}{G(t+1)}$$

$$- \frac{\rho(t)a(t+1)(\Delta^{\alpha} x(t+1))^{\gamma}\Delta G(t)}{G(t)G(t+1)}$$

$$\leq -k\rho(t)q(t) - \rho(t)\frac{p(t)(\Delta^{\alpha} x(t))^{\gamma}}{G(t)}$$

$$+\frac{\Delta\rho(t)}{\rho(t+1)}w(t+1)$$

$$-\frac{\Gamma(1-\alpha)\rho(t)a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta^{\alpha}x(t)}{G(t)G(t+1)}. \tag{9}$$

But from (6), we have

$$a(t)(\Delta^{\alpha}x(t))^{\gamma} \geq a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}, \tag{10}$$

and thus from (9) and (10), we have

$$\Delta w(t) \leq -k\rho(t)q(t)+\frac{1}{\rho(t+1)}\left[\Delta\rho(t)-\frac{\rho(t)p(t)}{a(t)}\right]w(t+1)$$

$$-\frac{\Gamma(1-\alpha)\rho(t)a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta^{\alpha}x(t)}{G(t+1)^{2}}. \tag{11}$$

So that

$$\Delta w(t) \leq -k\rho(t)q(t)+\frac{\Psi(t)}{\rho(t+1)}w(t+1)$$

$$-\frac{\Gamma(1-\alpha)\rho(t)}{(\rho(t+1))^{2}(a(t))^{\frac{1}{\gamma}}(a(t+1))^{\frac{\gamma-1}{\gamma}}}w^{2}(t+1)\frac{1}{(\Delta^{\alpha}x(t+1))^{\gamma-1}}, \tag{12}$$

where $\Psi(t)=\frac{a(t)\Delta\rho(t)-\rho(t)p(t)}{a(t)}$. Since $a(t)(\Delta^{\alpha}x(t))^{\gamma}$ is a positive and non increasing sequence, there exists a $t_2 \geq t_1$ sufficiently large such that $a(t)(\Delta^{\alpha}x(t))^{\gamma} \leq \frac{1}{M}$ for some positive constant M and $t \geq t_2$, and hence by (10) we have $a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma} \leq \frac{1}{M}$, so that

$$\frac{1}{(\Delta^{\alpha}x(t+1))^{\gamma-1}} \geq (Ma(t+1))^{\frac{\gamma-1}{\gamma}}. \tag{13}$$

Substitute (13) in (12), we obtain

$$\Delta w(t) \leq -k\rho(t)q(t)+\frac{\Psi(t)}{\rho(t+1)}w(t+1)$$

$$-\frac{\Gamma(1-\alpha)(Ma(t+1))^{\frac{\gamma-1}{\gamma}}\rho(t)}{(\rho(t+1))^{2}(a(t))^{\frac{1}{\gamma}}(a(t+1))^{\frac{\gamma-1}{\gamma}}}w^{2}(t+1) \tag{14}$$

$$\Delta w(t) \leq -k\rho(t)q(t) + \frac{\Psi(t)}{\rho(t+1)} w(t+1)$$

$$- \frac{R(t)}{(\rho(t+1))^2} w^2(t+1), \tag{15}$$

where $R(t) = \dfrac{\Gamma(1-\alpha)M^{\frac{\gamma-1}{\gamma}}\rho(t)}{(a(t))^{\frac{1}{\gamma}}}$.

Let $m = 2$, $X = \dfrac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}$ and $Y = \dfrac{\Psi(t)}{2\sqrt{R(t)}}$, using Lemma 2.3, we get

$$2\left(\frac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}\right)\left(\frac{\Psi(t)}{2\sqrt{R(t)}}\right) - \left(\frac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}\right)^2 \leq \frac{\Psi^2(t)}{4R(t)}$$

from (15), we conclude that

$$\Delta w(t) \leq -k\rho(t)q(t) + \frac{\Psi^2(t)}{4R(t)}.$$

Summing the above inequality t_2 to t, we have

$$\sum_{s=t_2}^{t}\left(k\rho(s)q(s) - \frac{\Psi^2(s)}{4R(s)}\right) \leq w(t_2) - w(t+1) < w(t_2) < \infty, \quad \text{for} \quad t \geq t_2.$$

Letting $t \to \infty$. Then

$$\lim_{t\to\infty} \sup \sum_{s=t_2}^{t}\left(k\rho(s)q(s) - \frac{\Psi^2(s)}{4R(s)}\right) < \infty,$$

which contradicts (5). The proof is complete.

From Theorem 3.2, we can obtain different conditions for oscillation of (1) by different choices of $\rho(t)$. For instance, we may let $\rho(t) = t^{\lambda}$, $t \geq t_0$, and $\lambda \geq 1$, or, we may $\rho(t) = R(t, t_0) = \sum_{s=t_0}^{t-1} 1/a(s)$. By Theorem 3.2, we will obtain the

following corollary.

Corollary 3.3. *Suppose that* (H1), (H5) *and* (Ha) *hold. If there is* $\lambda \geq 1$ *such that*

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t} \left(ks^{\lambda} q(s) - \frac{\Psi^2(s)}{4R(s)} \right) = \infty,$$

where $R(t) = \dfrac{\Gamma(1-\alpha) M^{\frac{\gamma-1}{\gamma}} t^{\lambda}}{(a(t))^{\frac{1}{\gamma}}}$ *and* $\Psi(t) = \dfrac{a(t)\Delta t^{\lambda} - t^{\lambda} p(t)}{a(t)}$ *for every positive constant* M, *then every solution of* (1) *is oscillatory.*

Corollary 3.4. *Suppose that* (H1), (H5) *and* (Ha) *hold. If there is* $\lambda \geq 1$ *such that*

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t} \left(kR(s, t_0) q(s) - \frac{\Psi^2(s)}{4R(s)} \right) = \infty,$$

where $R(t) = \dfrac{\Gamma(1-\alpha) M^{\frac{\gamma-1}{\gamma}} R(t, t_0)}{(a(t))^{\frac{1}{\gamma}}}$ *and* $\Psi(t) = \dfrac{a(t)\Delta R(t, t_0) - R(t, t_0) p(t)}{a(t)}$ *for every positive constant* M, *then every solution of* (1) *is oscillatory.*

Theorem 3.5. *Suppose that* (H1), (H3), (H4) *and* (Ha) *hold. Furthermore, assume that there exists a positive sequence* $\rho(t)$ *such that for every positive constant* M,

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t} \left(\rho(s) q(s) - \frac{\Psi^2(s)}{4R(s)} \right) = \infty, \tag{16}$$

where $R(t) = \dfrac{\tau\Gamma(1-\alpha) M^{\frac{\gamma-1}{\gamma}} \rho(t)}{(a(t))^{\frac{1}{\gamma}}}$ *and* $\Psi(t) = \dfrac{a(t)\Delta\rho(t) - \rho(t) p(t)}{a(t)}$, *then every solution of* (1) *is oscillatory.*

Proof. Assume to the contrary that $x(t)$ is a non-oscillatory solution of (1) such that (6) holds for $t \geq t_1$. Define the sequence $w(t)$ by

$$w(t) = \rho(t)\left(\frac{a(t)(\Delta^{\alpha}x(t))^{\gamma}}{f(G(t))}\right), \quad t \geq t_1. \tag{17}$$

Then $w(t) > 0$ and

$$\Delta w(t) = a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta\left[\frac{\rho(t)}{f(G(t))}\right] + \frac{\rho(t)\Delta(a(t)(\Delta^{\alpha}x(t))^{\gamma})}{f(G(t))}. \tag{18}$$

Using (1)

$$\Delta w(t) = a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\left[\frac{f(G(t))\Delta\rho(t) - \rho(t)\Delta f(G(t))}{f(G(t+1))f(G(t))}\right]$$

$$- \rho(t)\left(\frac{\rho(t)(\Delta^{\alpha}x(t))^{\gamma} + q(t)f(G(t))}{f(G(t))}\right).$$

In view of $(\mathrm{H3})$ and (18), we have

$$\Delta w(t) \leq -\rho(t)q(t) - \rho(t)\frac{p(t)(\Delta^{\alpha}x(t))^{\gamma}}{f(G(t))} + \frac{\Delta\rho(t)}{\rho(t+1)}w(t+1)$$

$$- \frac{\tau\Gamma(1-\alpha)\rho(t)a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta^{\alpha}x(t)}{f(G(t))f(G(t+1))}. \tag{19}$$

But from (6), we have

$$a(t)(\Delta^{\alpha}x(t))^{\gamma} \geq a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma} \tag{20}$$

and thus from (19) and (20), we have

$$\Delta w(t) = -\rho(t)q(t) - \rho(t)\frac{p(t)a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}}{a(t)f(G(t+1))} + \frac{\Delta\rho(t)}{\rho(t+1)}w(t+1)$$

$$- \frac{\tau\Gamma(1-\alpha)\rho(t)a(t+1)(\Delta^{\alpha}x(t+1))^{\gamma}\Delta^{\alpha}x(t)}{f(G(t+1))^2}. \tag{21}$$

So that

$$\Delta w(t) = -\rho(t)q(t) + \frac{\Psi(t)}{\rho(t+1)} w(t+1)$$

$$- \frac{\tau\Gamma(1-\alpha)\rho(t)a^{\frac{1}{\gamma}}(t+1)}{(\rho(t+1))^2 a^{\frac{1}{\gamma}}(t)} w^2(t+1) \frac{1}{(\Delta^\alpha x(t+1))^{\gamma-1}}, \tag{22}$$

where $\Psi(t) = \frac{a(t)\Delta\rho(t) - \rho(t)p(t)}{a(t)}$. Since $a(t)(\Delta^\alpha x(t))^\gamma$ is a positive and non increasing sequence, there exists a $t_2 \geq t_1$ sufficiently large such that $a(t)(\Delta^\alpha x(t))^\gamma \leq \frac{1}{M}$ for some positive constant M and $t \geq t_2$, and hence by (20) we have $a(t+1)(\Delta^\alpha x(t))^\gamma \leq \frac{1}{M}$, so that

$$\frac{1}{(\Delta^\alpha x(t+1))^{\gamma-1}} \geq (Ma(t+1))^{\frac{\gamma-1}{\gamma}}. \tag{23}$$

Substitute from (23) in (22), we obtain

$$\Delta w(t) \leq -\rho(t)q(t) + \frac{\Psi(t)}{\rho(t+1)} w(t+1)$$

$$- \frac{\tau\rho(t)\Gamma(1-\alpha)M^{\frac{\gamma-1}{\gamma}}}{(a(t))^{\frac{1}{\gamma}}(\rho(t+1))^2} w^2(t+1) \tag{24}$$

$$\leq -\rho(t)q(t) + \frac{\Psi(t)}{\rho(t+1)} w(t+1) - \frac{R(t)}{(\rho(t+1))^2} w^2(t+1), \tag{25}$$

where $R(t) = \frac{\Gamma(1-\alpha)M^{\frac{\gamma-1}{\gamma}}\rho(t)}{(a(t))^{\frac{1}{\gamma}}}$.

Let $m = 2$, $X = \frac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}$ and $Y = \frac{\Psi(t)}{2\sqrt{R(t)}}$, using Lemma 2.3 we get

$$2\left(\frac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}\right)\left(\frac{\Psi(t)}{2\sqrt{R(t)}}\right) - \left(\frac{\sqrt{R(t)}w(t+1)}{\rho(t+1)}\right)^2 \leq \left(\frac{\Psi(t)}{2\sqrt{R(t)}}\right)^2$$

from (25), we conclude that

$$\Delta w(t) \leq -\rho(t)q(t) + \frac{\Psi^2(t)}{4R(t)}.$$

Summing the above inequality t_2 to t, we have

$$\sum_{s=t_2}^{t} (\rho(s)q(s)) - \frac{\Psi^2(s)}{4R(s)} \leq w(t_2) - w(t+1) < w(t_2) < \infty, \quad \text{for} \quad t \geq t_2.$$

Letting $t \to \infty$. Then

$$\lim_{t\to\infty} \sup \sum_{s=t_2}^{t} \left(\rho(s)q(s) - \frac{\Psi^2(s)}{4R(s)} \right) < \infty,$$

which contradicts (16). The proof is complete.

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