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OSTROWSKI-TYPE INEQUALITIES VIA h- CONVEX STOCHASTIC PROCESSES

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Abstract

In this paper, we prove Ostrowski's type inequalities for stochastic process whose first derivatives are h- convex, with h super-additive and super-multiplicative functions.

1. Introduction

The origins of the stochastic processes study come from the endings of 30's and the stochastic convexity appeared in [17], where Nikodem introduced this notion and some properties of convex stochastic processes were proved based on the definition of additive processes introduced by Nagy in 1974 (see [16]).

Recently, some inequalities for convex stochastic processes have been established. Among them, N. Merentes et al., stated Jensen and Hermite-Hadamard type inequalities in [5]. Also, error estimations of a Hermite-Hadamard inequality type for stochastic processes were given in [15], and a representation of the

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Simpson's inequality for stochastic processes using s- convexity and quasi-convexity was presented in [18]. In addition, in [19] several inequalities for Ostrowski-type through convex, s- convex and quasi-convex stochastic processes were introduced. In this research, the inequalities are related to the left hand side of Hadamard inequality.

Following this line of investigation, we consider other important inequality: the Ostrowski's inequality [20], which for a differentiable function on I°, $f : I \to \mathbb{R}$ establishes that if $a, b \in I^{\circ}$ with $a < b$ and $f' : (a, b) \to \mathbb{R}$ is integrable on (a, b) and bounded, i.e., $\| f' \|_{\infty} := \sup_{t \in (a, b)} | f'(t) | < \infty$, the following expression holds:

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \| f' \|_{\infty} (b-a) \left[\frac{1}{4} + \frac{\left(t - \frac{a+b}{2} \right)}{(b-a)^2} \right], \tag{1.1}$$

for all $t \in [a, b]$.

Many studies have been done about the Ostrowski's inequality for functions. Dragomir and Wang in [13-14] extended the inequality (1.1) for absolutely continuous functions and applied the extended result to numerical quadrature rules and to the estimation of error bounds for special means. The interested reader can see other generalizations in [1]-[14], [21-22].

In this work, we present a counterpart of the research made by Tunç in [24] for stochastic processes to estimate different refinements of the weighted difference in absolute value between the mean integrals of a stochastic process X and its first derivative, considering the h- convexity condition on the first derivative in absolute value.

2. Ostrowski-type Inequalities for h- convex Stochastic Processes

First, we would like to recall some useful definitions. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process such that $\mathbb{E}[X(t)]^2 < \infty$ for all $t \in I$, where $\mathbb{E}[X(t)]$ denotes the expectation value of $X(t, \cdot)$. The stochastic process X is called

(1) *mean-square differentiable* in I, if there exists a stochastic process X' (the derivative of X) such that for all $t_0 \in I$, we have

$$\lim_{t \to t_0} E\left[\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\right]^2 = 0.$$

(2) *mean-square integrable* on $[a, b] \subseteq I$, if there exists a random variable Y such that for all normal *sequence* of partitions of the interval $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$, we have

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} X(\Theta_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0.$$

The random variable $Y : \Omega \to \mathbb{R}$ is called the *mean-square integral* of the process X on $[a, b]$. In such case, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds, \quad \text{(a.e.)}.$$

Definition and basic properties of the mean-square derivative and mean-square integral can be read in [23].

Fixed $h : (0, 1) \to \mathbb{R}$, we say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is an h-*convex stochastic process* if, for every $t_1, t_2 \in I$, $\lambda \in (0, 1)$, the following inequality is satisfied

$$X(\lambda t_1 + (1 - \lambda)t_2, \cdot) \le h(\lambda) X(t_1, \cdot) + h(1 - \lambda) X(t_2, \cdot), \quad \text{(a.e.)} \tag{2.1}$$

Taking h equals to the identity, the definition of stochastic h- convexity matches with the stochastic convexity one. In [5], some interesting properties of h-*convex* stochastic processes were proved.

If in (2.1), the reversed inequality holds, the stochastic process is h-*concave*.

The equality stated in the following lemma is fundamental in this paper.

Lemma 2.1. *Let* $X : I \times \Omega \to \mathbb{R}$ *be a stochastic process mean-square differentiable on* I°. *If* X' *is mean-square integrable on* $[a, b]$, *where* $a, b \in I$ *with* $a < b$, *then the following equality holds*:

$$X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)\,du$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 tX'(tx + (1-t)a, \cdot)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 tX'(tx + (1-t)b, \cdot)\,dt, \quad (a.e.)$$

for each $x \in [a, b]$.

Proof. Changing variables, $u = yt + (1-y)a$ and $w = yt + (1-y)b$ and integrating by parts, we have

$$\frac{(t-a)^2}{b-a}\int_0^1 yX'(yt + (1-y)a, \cdot)\,dy - \frac{(b-t)^2}{b-a}\int_0^1 yX'(yt + (1-y)b, \cdot)\,dy$$

$$= \frac{(t-a)^2}{b-a}\int_a^t \frac{(u-a)}{(t-a)} X'(u, \cdot)\frac{du}{(t-a)} - \frac{(b-t)^2}{b-a}\int_t^b \frac{(b-w)}{(b-t)} X'(w, \cdot)\frac{dw}{(b-t)}$$

$$= \frac{1}{b-a}\int_a^t (u-a)X'(u, \cdot)\,du - \frac{1}{b-a}\int_t^b (b-w)X'(w, \cdot)\,dw$$

$$= \frac{1}{b-a}\left[(t-a)X(t, \cdot) - \int_a^t X(u, \cdot)\,du\right]$$

$$+ \frac{1}{b-a}\left[(b-t)X(t, \cdot) - \int_t^b X(w, \cdot)\,dw\right]$$

$$= \frac{1}{b-a}\left[(t-a)X(t, \cdot) - \int_a^t X(u, \cdot)\,du + (b-t)X(t, \cdot) - \int_t^b X(w, \cdot)\,dw\right]$$

$$= \frac{1}{b-a}\left[X(t, \cdot)(b-a) - \int_a^b X(u, \cdot)\,du\right]$$

$$= X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)\,du, \quad \text{(a.e)}$$

so proof is completed.

Now, we are ready to present some Ostrowski-type inequalities for h- convex stochastic processes.

Theorem 2.2. *Let* $h : (0, 1) \to \mathbb{R}$ *be a nonnegative and supermultiplicative function such that* $h(\alpha) > \alpha$ *for every* α *and let* $X : I \times \Omega \to \mathbb{R}$ *be a mean-square stochastic process such that* X' *is mean-square integrable on* $[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|X'|$ *is an* h- *convex stochastic process on* I *and* $|X'(t, \cdot)| \leq M$ *for every* t, *then*

$$\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right|$$

$$\leq \frac{M[(t-a)^2 + (b-t)^2]}{b-a} \int_0^1 [h(t^2) + h(h - t^2)] dt \tag{2.2}$$

for each $t \in [a, b]$.

Proof. By Lemma 2.1 and since $|X'|$ is h- convex, then we can write

$$\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right|$$

$$\leq \frac{(t-a)^2}{b-a} \int_0^1 y|X'(yt + (1-y)a, \cdot)| dy + \frac{(b-t)^2}{b-a} \int_0^1 y|X'(yt + (1-y)b, \cdot)| dy$$

$$\leq \frac{(t-a)^2}{b-a} \int_0^1 y[h(y)|X'(t, \cdot)| + h(1-y)|X'(a, \cdot)|] dy$$

$$+ \frac{(b-t)^2}{b-a} \int_0^1 y[h(y)|X'(t, \cdot)| + h(1-y)|X'(b, \cdot)|] dy$$

$$\leq \frac{M(t-a)^2}{b-a} \int_0^1 [yh(y) + yh(1-y)] dy + \frac{M(b-t)^2}{b-a} \int_0^1 [yh(y) + yh(1-y)] dy$$

$$\leq M \left[\frac{(t-a)^2 + (b-t)^2}{b-a} \right] \int_0^1 [(h(y))^2 + h(y) h(1-y)]$$

$$= M \left[\frac{(t-a)^2 + (b-t)^2}{b-a} \right] \int_0^1 [h(y)^2 + h(y - y^2)] dy$$

and the proof is complete. □

Corollary 2.3. *If we choose* $h(y) = y^s$, *in* (2.2), *then*

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{M[(t-a)^2 + (b-t)^2]}{b-a}\int_0^1 [t^{2s} + t^s(1-t)^s]dt$$

$$\leq \frac{M[(t-a)^2 + (b-t)^2]}{b-a}\left[\frac{1}{2s+1} + \frac{\Gamma(s+1)\Gamma(s+1)}{\Gamma(2s+2)}\right]$$

$$\leq \frac{M[(t-a)^2 + (b-t)^2]}{b-a}\left[\frac{\Gamma(2s+2) + (2s+1)(\Gamma(s+1))^2}{2s+1\Gamma(2s+2)}\right]$$

$$\leq \frac{M[(t-a)^2 + (b-t)^2]}{b-a}\left[\frac{\Gamma(2s+1) + s^2(\Gamma(s))^2}{(2s+1)\Gamma(2s+1)}\right],$$

where Γ *denotes the Gamma function.*

One of the important result is given in the following theorem.

Theorem 2.4. *Let* $h : (0, 1) \to \mathbb{R}$ *be a nonnegative and superadditive function such that* $h(t) \geq t$ *for every* t *and let* $X : I \times \Omega \to \mathbb{R}$ *be a mean-square differentiable stochastic process such that* X' *is a mean-square integrable on* $[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|X'|^q$ *is an* h-*convex stochastic process on* $[a, b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $|X'(t)| \leq M$ *for every* $t \in [a, b]$, *then*

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{Mh^{1/q}(1)}{b-a}\left(\int_0^1 h(y^p)dt\right)^{1/p}[(t-a)^2 + (b-t)^2], \quad (a.e),$$

for each $t \in [a, b]$.

Proof. Suppose that $p > 1$. From Lemma 2.1 and using Hölder's inequality, we can write

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{(t-a)^2}{b-a}\int_0^1 y|X'(yt+(1-y)a, \cdot)|dy + \frac{(b-t)^2}{b-a}\int_0^1 y|X'(yt+(1-y)b, \cdot)|dy$$

$$\leq \frac{(t-a)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 |X'(ty+(1-y)a, \cdot)|^q dy\right)^{1/q}$$

$$+ \frac{(b-t)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 |X'(ty+(1-y)b, \cdot)|^q dy\right)^{1/q}.$$

Since $|X'|^q$ is h- convex, by using the properties of h- convexity in the assumptions, we have

$$\int_0^1 |X'(ty+(1-y)a, \cdot)|^q dy$$

$$\leq \int_0^1 [h(y)|X'(t, \cdot)|^q + h(1-y)|X'(a, \cdot)|^q]dy$$

$$\leq M^q \int_0^1 [h(y)+h(1-y)]dy$$

$$\leq M^q \int_0^1 h(1)dt$$

$$= M^q h(1).$$

Similarly

$$\int_0^1 |X'(ty+(1-y)b, \cdot)|^q dy = M^q h(1), \quad \text{(a.e.)}$$

and

$$\int_0^1 y^p dy \leq \int_0^1 h(y^p)dy.$$

Therefore, we obtain

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_0^1 X(u, \cdot)du \right|$$

$$\leq Mh^{1/q}(1)\frac{(t-a)^2}{b-a}\left(\int_0^1 h(y^p)dy\right)^{1/p} + Mh^{1/q}(1)\frac{(b-t)^2}{b-a}\left(\int_0^1 h(y^p)dy\right)^{1/p}$$

$$= \frac{Mh^{1/q}(1)}{b-a}\left(\int_0^1 h(y^p)dy\right)^{1/p}[(t-a)^2 + (b-t)^2], \quad \text{(a.e.)}.$$

The proof is completed. □

Corollary 2.5. *Choosing* $h(t) = t^n$ $(n \in \mathbb{N},\, n \geq 2)$ *in* (2.3), *we have*

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{M}{b-a}\left(\frac{1}{np+1}\right)^{1/p}[(t-a)^2 + (b-t)^2], \quad (a.e.). \tag{2.4}$$

As we know, h- convex stochastic processes are a generalization of convex stochastic processes [5]. In this sense, it is usual to obtain weaker results once compared with inequalities in referenced studied, because the inequalities written here were considered to be more general than the above mentioned classed, and it was taken into account to be super-multiplicative or super-additive functions. In this case, the right side of inequality may be greater.

A new approach to an h-convex stochastic process is given in the following result.

Theorem 2.6. *Let* $h : (0, 1) \to \mathbb{R}$ *be a nonnegative and supermultiplicative function such that* $h(\alpha) \geq \alpha$ *for every* α *and let* $X : I \times \Omega \to \mathbb{R}$ *be a mean-square differentiable stochastic processes on* I° *such that* X' *is mean square integrable on* $[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|X'|^q$ *is an* h- *convex stochastic process on* $[a, b]$, $q \geq 1$, *and* $|X'(t, \cdot)| \leq M$ *for every* t, *then*

$$\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right|$$

$$\le \frac{\sqrt[q]{2}M}{2(b-a)}[(t-a)^2 + (b-t)^2]\left(\int_0^1 (h(y^2) + h(y - y^2))dy\right)^{1/q}, \quad (a.e.), \qquad (2.5)$$

for each $t \in [a, b]$.

Proof. Suppose that $q \ge 1$. From Lemma 2.1, and using the power mean in equality, we have

$$\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right|$$

$$\le \frac{(t-a)^2}{b-a}\int_0^1 y|X'(yt + (1-y)a,\cdot)|dt + \frac{(b-t)^2}{b-a}\int_0^1 y|X'(yt + (1-y)b,\cdot)|dy$$

$$\le \frac{(t-a)^2}{b-a}\left(\int_0^1 ydy\right)^{1-\frac{1}{q}}\left(\int_0^1 y|X'(yt + (1-y)a,\cdot)|^q dy\right)^{\frac{1}{q}}$$

$$+ \frac{(b-t)^2}{b-a}\left(\int_0^1 ydy\right)^{1-\frac{1}{q}}\left(\int_0^1 y|X'(yt + (1-y)b,\cdot)|^q dy\right)^{\frac{1}{q}}.$$

Now, since $|X'|^q$ is h- convex, we have

$$\int_0^1 y|X'(yt + (1-y)a,\cdot)|^q dy$$

$$\le \int_0^1 [yh(y)|X'(t,\cdot)|^q + yh(1-y)|X'(a,\cdot)|^q dy]$$

$$= \int_0^1 yh(y)|X'(t,\cdot)|^q dy + \int_0^1 yh(1-y)|X(a,\cdot)|^q dy$$

$$\le |X'(t,\cdot)|^q \int_0^1 h(y)h(y)dy + |X(a,\cdot)|^q \int_0^1 h(y)h(1-y)dy$$

$$\le M^q \int_0^1 h(y^2)dy + M^q \int_0^1 h(y - y^2)dy$$

$$= M^q \left[\int_0^1 h(y^2)dy + \int_0^1 h(y - y^2)dy\right].$$

In a similar way, we obtain

$$\int_0^1 y| X'(yt + (1 - y)b, \cdot)|^q dy \leq M^q \left[\int_0^1 h(y^2)dy + \int_0^1 h(y - y^2)dy\right].$$

Therefore,

$$\left| X'(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{(t-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(M^q \int_0^1 h(y^2) + h(y - y^2)dy\right)^{1/q}$$

$$+ \frac{(b-t)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(M^q \int_0^1 h(y^2) + h(y - y^2)dy\right)^{1/q}$$

$$= M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 h(y^2) + h(y - y^2)dy\right)^{1/q}\left(\frac{(t-a)^2 + (b-t)^2}{(b-a)}\right)$$

$$= M\sqrt[q]{2}\left(\int_0^1 h(y^2) + h(y - y^2)dy\right)^{1/q}\left(\frac{(t-a)^2 + (b-t)^2}{2(b-a)}\right)$$

which completes the proof. □

Remark 2.7. (i) In the above inequalities, is possible to establish several midpoint-type inequalities by letting $t = \frac{a+b}{2}$.

(ii) In Theorem 2.6, if we choose

(1) $t = \frac{a+b}{2}$, we obtain

$$\left| X'\left(\frac{a+b}{2}, \cdot\right) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right| \leq \frac{\sqrt[q]{2}M(b-a)}{4}\left(\int_0^1 (h(y^2) + h(y - y^2))dy\right)^{1/q}.$$

(2) $t = a$, then we obtain

$$\left| X'(a, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right| \leq \frac{\sqrt[q]{2}M(b-a)}{2}\left(\int_0^1 (h(y^2) + h(y - y^2))dy\right)^{1/q}.$$

(3) $t = b$, then we obtain

$$\left| X'(b, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right| \leq \frac{\sqrt[q]{2}M(b-a)}{2}\left(\int_0^1 (h(y^2) + h(y - y^2))dy\right)^{1/q}.$$

The following result holds for h- concave process stochastic.

Theorem 2.8. *Let $h : (0, 1) \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative and superadditive function such that $h(y) \geq y$ for every y and let $X : I \times \Omega \to \mathbb{R}$ be a mean-square differentiable on $I°$ stochastic process such that X' is mean-square integrable, where $a, b \in I$ with $a > b$. If $| X' |^q$ is an h- concave stochastic process on $[a, b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{1}{\sqrt[q]{2}(p+1)^{1/p} h^{1/q}\left(\frac{1}{2}\right)}\left[\frac{(t-a)^2}{b-a}\left| X\left(\frac{a+t}{2}, \cdot\right)\right| + \frac{(b-t)^2}{b-a}\left| X\left(\frac{t+b}{2}, \cdot\right)\right|\right],$$

for each $t \in [a, b]$.

Proof. Suppose $p > 1$. From Lemma 2.1 and using Hölder's inequality, we have

$$\left| X(t, \cdot) - \frac{1}{b-a}\int_a^b X(u, \cdot)du \right|$$

$$\leq \frac{(t-a)^2}{b-a}\int_0^1 y| X'(yt + (1-y)a, \cdot)|dy + \frac{(b-t)^2}{b-a}\int_0^1 y| X'(yt + (1-y)b, \cdot)|dy$$

$$\leq \frac{(t-a)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 | X'(yt + (1-y)a, \cdot)|^q dy\right)^{1/q}$$

$$+ \frac{(b-t)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 | X'(yt + (1-y)b, \cdot)|^q dy\right)^{1/q}.$$

Now, since $|X'|^q$ is h- concave, and using Theorem 2.10, proved in [5], we have

$$\left(\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy\right) \leq \frac{1}{2h\left(\frac{1}{2}\right)}\left|X\left(\frac{a+t}{2},\cdot\right)\right|^q$$

and

$$\left(\int_0^1 |X'(yt+(1-y)b,\cdot)|^q dy\right) \leq \frac{1}{2h\left(\frac{1}{2}\right)}\left|X'\left(\frac{t+b}{2},\cdot\right)\right|^q.$$

Furthermore

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_0^1 X(u,\cdot)\,du\right|$$

$$\leq \frac{(t-a)^2}{b-a}\left(\frac{y^{p+1}}{p+1}\bigg|_0^1\right)^{1/p}\left(\frac{1}{2h\left(\frac{1}{2}\right)}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|^q\right)^{1/q}$$

$$+\frac{(b-t)^2}{b-a}\left(\frac{y^{p+1}}{p+1}\bigg|_0^1\right)^{1/p}\left(\frac{1}{2h\left(\frac{1}{2}\right)}\left|X'\left(\frac{b+t}{2},\cdot\right)\right|^q\right)^{1/q}$$

$$=\frac{(t-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{2h\left(\frac{1}{2}\right)}\right)^{1/q}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|$$

$$+\frac{(b-t)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{2h\left(\frac{1}{2}\right)}\right)^{1/q}\left|X'\left(\frac{b+t}{2},\cdot\right)\right|$$

$$=\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{2h\left(\frac{1}{2}\right)}\right)^{1/q}\left[\frac{(t-a)^2}{b-a}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|+\frac{(b-t)^2}{b-a}\left|X'\left(\frac{b+t}{2},\cdot\right)\right|\right]$$

$$= \frac{1}{(p+1)^{1/p}} \frac{1}{\sqrt[q]{2} h^{1/q}\left(\frac{1}{2}\right)} \left[\frac{(t-a)^2}{b-a} \left| X'\left(\frac{a+t}{2}, \cdot\right) \right| + \frac{(b-t)^2}{b-a} \left| X'\left(\frac{b+t}{2}, \cdot\right) \right| \right]$$

and the proof is complete. □

A midpoint-type inequality for stochastic process whose derivatives in absolute value are h-concave may be established from the above result as follows.

Corollary 2.9. *In Theorem* 2.8, *if we choose* $t = \frac{a+b}{2}$, *then*

$$\left| X\left(\frac{a+b}{2}, \cdot\right) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right|$$

$$\leq \frac{b-a}{\sqrt[q]{2^{2q+1}} (p+1)^{\frac{1}{p}} h^{\frac{1}{q}}\left(\frac{1}{2}\right)} \left[\left| X'\left(\frac{3a+b}{4}, \cdot\right) \right| + \frac{(b-t)^2}{b-a} \left| X'\left(\frac{t+b}{2}, \cdot\right) \right| \right].$$

For instance, if $h(y) = y$, then we obtain

$$\left| X\left(\frac{a+b}{2}, \cdot\right) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right|$$

$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left[\left| X'\left(\frac{3a+b}{4}, \cdot\right) \right| + \left| X'\left(\frac{a+3b}{4}\right) \right| \right],$$

where $|X'|^q$ is an h- concave process stochastic on $[a, b]$, $p, q > 1$.

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