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SEQUENCES OF DIOPHANTINE TRIPLES

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Abstract

This paper concerns with the study of constructing sequences of Diophantine triples (a, b, c) such that the product of any two elements of the set added by a polynomial with integer coefficient is a perfect square.

1. Introduction

The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets have been studied by Diophantus. A set of m positive integers $\{a_1, a_2, a_3, \ldots, a_m\}$ is said to have the *property* $D(n)$, $n \in z - \{0\}(a_i * a_j) + n$, it is a perfect square for all $1 \leq i \leq j \leq m$ and such a set is called a *Diophantine m-tuple* with the property $D(n)$.

Many mathematicians considered the construction of different formulations of Diophantine triples with the property $D(n)$ for any arbitrary integer n and also, for

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any linear polynomials in n. In this context, one may refer [2-19] for an extensive review of various problems on Diophantine triples. This paper aims at constructing sequences of Diophantine triples where the product of any two members of the triple with the polynomial with integer coefficients satisfies the required property.

2. Method of Analysis

Sequence I

An attempt is made to form a sequence of Diophantine triples (a, b, c), (b, c, d), $(c, d, e), \ldots$ with the property $D(1 + 3^n)$.

Case I

Let $a = 3^n$ and $b = 3^n + 1$.

Let c be any non-zero integer.

Consider

$$ac + 1 + 3^n = p^2$$

which yields

$$(3^n)c + 1 + 3^n = p^2. \tag{1}$$

Also,

$$bc + 1 + 3^n = q^2$$

gives

$$(3^n + 1)c + 1 + 3^n = q^2. \tag{2}$$

Using some algebra,

$$(3^n + 1)p^2 - 3^n q = (1 + 3^n). \tag{3}$$

Using the linear transformations

$$p = X + 3^n T,$$

$$q = X + (3^n + 1)T$$

and $T = 1$, we have

$$X = 3^n + 1 \quad \text{and} \quad p = 2 \cdot 3^n + 1.$$

From (1),

$$c = 4 \cdot 3^n + 3.$$

Hence (a, b, c) is the Diophantine triple with the property $D(1 + 3^n)$.

Case II

Let $b = 3^n + 1$ and $c = 4 \cdot 3^n + 3.$

Let d be any non-zero integer.

Consider

$$bd + 1 + 3^n = \beta^2,$$

$$cd + 1 + 3^n = \gamma^2.$$

On simplification, we have

$$(3^n + 1)d + (1 + 3^n) = \beta^2, \tag{4}$$

$$(4 \cdot 3^n + 3)d + (1 + 3^n) = \gamma^2. \tag{5}$$

Using some algebra,

$$c\beta^2 - b\gamma^2 = (c - b)(1 + 3^n).$$

Using the linear transformations

$$\beta = X + bT \quad \text{and} \quad \gamma = X + cT$$

and $T = 1$, we have

$$X = 2(3^n + 1) \quad \text{and} \quad \beta = 3 \cdot 3^n + 3.$$

From (4),

$$d = 9 \cdot 3^n + 8.$$

Hence (b, c, d) is the Diophantine triple with the property $D(1+3^n)$.

Case III

Let $c = 4 \cdot 3^n + 3$ and $d = 9 \cdot 3^n + 8$.

Let e be any non-zero integer.

Consider,

$$ce + 1 + 3^n = \delta^2, \tag{6}$$

$$de + 1 + 3^n = \theta^2. \tag{7}$$

Using some algebra,

$$d\delta^2 - c\theta^2 = (d - c)(1 + 3^n).$$

Using the linear transformations,

$$\delta = X + cT \quad \text{and} \quad \theta = X + dT$$

and $T = 1$, we have

$$X = 6 \cdot 3^n + 5 \quad \text{and} \quad \delta = 10 \cdot 3^n + 8.$$

From (5),

$$e = 25 \cdot 3^n + 21.$$

Thus (c, d, e) forms a Diophantine triple with the property $D(1+3^n)$.

From all the above cases, $(a, b, c), (b, c, d), (c, d, e), \ldots$ will form a sequence of Diophantine triples.

Some numerical examples are tabulated

n	(a, b, c)	(b, c, d)	(c, d, e)	$D(1+3^n)$
0	(1, 2, 7)	(2, 7, 17)	(7, 17, 46)	$D(2)$
1	(3, 4, 15)	(4, 15, 35)	(15, 35, 96)	$D(4)$
2	(9, 10, 39)	(10, 39, 89)	(39, 89, 246)	$D(10)$
3	(27, 28, 111)	(28, 111, 251)	(11, 251, 696)	$D(28)$

Sequence II

Deriving another sequence of Diophantine triples $(a, b, c), (b, c, d), (c, d, e), \ldots$ with the property $D(1 + 3^n)$.

Case I

Let $a = 3^n$ and $b = 3^n + 3$.

Let c be any non-zero integer.

Consider

$$ac + 1 - 3^n = p^2 \tag{8}$$

which yields

$$(3^n)c + 1 - 3^n = p^2,$$

$$bc + 1 - 3^n = q^2$$

gives

$$(3^n + 3)c + 1 - 3^n = q^2.$$

Using some algebra,

$$(3^n + 3)p^2 - 3^n q^2 = 3(1 - 3^n).$$

Using the linear transformations

$$p = X + 3^n T,$$

$$q = X + (3^n + 3)T$$

and $T = 1$, we have $X = 3^n + 1$ and $p = 2 \cdot 3^n + 1$.

From (8),

$$c = 4 \cdot 3^n + 5.$$

Hence (a, b, c) is the Diophantine triple with the property $D(1 - 3^n)$.

Case II

Let $b = 3^n + 3$ and $c = 4 \cdot 3^n + 5.$

Let d be any non-zero integer.

Consider

$$bd + 1 - 3^n = \beta^2,$$

$$cd + 1 - 3^n = \gamma^2. \tag{9}$$

On simplification, we have

$$(3^n + 3)d + (1 - 3^n) = \beta^2,$$

$$(4 \cdot 3^n + 5)d + (1 - 3^n) = \gamma^2.$$

Using some algebra,

$$c\beta^2 - b\gamma^2 = (c - b)(1 + 3^n).$$

Using the linear transformations

$$\beta = X + bT \quad \text{and} \quad \gamma = X + cT$$

and $T = 1,$ we have

$$X = 2(3^n + 2) \quad \text{and} \quad \beta = 3 \cdot 3^n + 7.$$

From (9),

$$d = 9 \cdot 3^n + 16.$$

Hence (b, c, d) is the Diophantine triple with the property $D(1 - 3^n).$

Case III

Let $c = 4 \cdot 3^n + 5$ and $d = 9 \cdot 3^n + 16.$

Let e be any non-zero integer.

Consider

$$ce + 1 - 3^n = \delta^2,$$

$$de + 1 - 3^n = \theta^2. \tag{10}$$

Using some algebra,

$$d\delta^2 - c\theta^2 = (d - c)(1 - 3^n).$$

Using the linear transformations,

$$\delta = X + cT \quad \text{and} \quad \theta = X + dT$$

and $T = 1$, we have

$$X = 6.3^n + 9 \quad \text{and} \quad \delta = 10 \cdot 3^n + 14.$$

From (10),

$$e = 25 \cdot 3^n + 39.$$

Thus (c, d, e) forms a Diophantine triple with the property $D(1 - 3^n)$.

Case IV

Let $d = 9 \cdot 3^n + 16$ and $e = 25 \cdot 3^n + 39$.

Let *f* be any non-zero integer.

Consider

$$df + 1 - 3^n = \alpha^2,$$

$$ef + 1 - 3^n = \lambda^2. \tag{11}$$

Using some algebra,

$$e\alpha^2 - d\lambda^2 = (e - d)(1 - 3^n).$$

Using the linear transformations,

$$\alpha = X + dT \quad \text{and} \quad \lambda = X + eT$$

and $T = 1$, we have

$$X = 15 \cdot 3^n + 25 \quad \text{and} \quad \alpha = 24 \cdot 3^n + 41.$$

From (11),

$$f = 64 \cdot 3^n + 105.$$

Thus (d, e, f) forms a Diophantine triple with the property $D(1-3^n)$.

From all the above cases, $(a, b, c), (b, c, d), (c, d, e), (d, e, f), \ldots$ will form a sequence of Diophantine triples.

Some numerical examples are tabulated

n	(a, b, c)	(b, c, d)	(c, d, e)	$D(1-3^n)$
1	(3, 6, 17)	(6, 17, 43)	(17, 43, 114)	$D(-2)$
2	(9, 12, 41)	(12, 41, 97)	(41, 97, 264)	$D(-8)$
3	(27, 30, 113)	(30, 113, 259)	(113, 259, 714)	$D(-26)$
4	(81, 84, 329)	(84, 329, 745)	(329, 745, 2064)	$D(-80)$

Sequence III

Forming a sequence of Diophantine triples $(a, b, c), (b, c, d), (c, d, e), \ldots$ with the property $D(8 \cdot 2^{2n})$.

Case I

Let $a = 2^{2n} - 2^{n+1} - 1$ and $b = 2^{2n} + 2^{n+1} - 1$.

Let c be any non-zero integer.

Consider

$$ac + 8 \cdot 2^{2n} = p^2$$

which yields

$$(2^{2n} - 2^{n+1} - 1)c + 8 \cdot 2^{2n} = p^2,$$

$$bc + 8 \cdot 2^{2n} = q^2 \tag{12}$$

gives

$$(2^{2n} + 2^{n+1} - 1)c + 8 \cdot 2^{2n} = q^2.$$

Using some algebra,

$$(2^{2n} + 2^{n+1} - 1)p^2 - (2^{2n} - 2^{n+1} - 1)q^2 = 16 \cdot 2^{2n} \cdot 2^{2n+1}.$$

Using the linear transformations

$$p = X + (2^{2n} - 2^{n+1} - 1)T,$$

$$q = X + (2^{2n} + 2^{n+1} - 1)T$$

and $T = 1$, we have

$$X = 2^{2n} + 1 \quad \text{and} \quad p = 2 \cdot 2^{2n} - 2^{n+1}.$$

From (12),

$$c = 4 \cdot 2^{2n}.$$

Hence (a, b, c) is the Diophantine triple with the property $D(8 \cdot 2^{2n})$.

Case II

Let $b = 2^{2n} + 2^{n+1} - 1$ and $c = 4 \cdot 2^{2n}$.

Let d be any non-zero integer.

Consider

$$bd + 8 \cdot 2^{2n} = \beta^2,$$

$$cd + 8 \cdot 2^{2n} = \gamma^2. \qquad (13)$$

Using some algebra,

$$c\beta^2 - b\gamma^2 = (c - b)(8 \cdot 2^{2n}).$$

Using the linear transformations

$$\beta = X + bT \quad \text{and} \quad \gamma = X + cT$$

and $T = 1$, we have

$$X = 2(2^{2n} + 2^n) \text{ and } \beta = 3 \cdot 2^{2n} + 4 \cdot 2^n - 1.$$

From (13),

$$d = 9 \cdot 2^{2n} + 6 \cdot 2^2 - 1.$$

Hence (b, c, d) is the Diophantine triple with the property $D(8 \cdot 2^{2n})$.

Case III

Let $c = 4 \cdot 2^{2n}$ and $d = 9 \cdot 2^{2n} + 6 \cdot 2^n - 1.$

Let e be any non-zero integer.

Consider

$$ce + 8 \cdot 2^{2n} = \delta^2,$$

$$de + 8 \cdot 2^{2n} = \theta^2. \tag{14}$$

Using some algebra,

$$d\delta^2 - c\theta^2 = (d - c)(8 \cdot 2^{2n}).$$

Using the linear transformations,

$$\delta = X + cT \quad \text{and} \quad \theta = X + dT$$

and $T = 1,$ we have

$$X = 6 \cdot 2^{2n} + 2 \cdot 2^n \quad \text{and} \quad \delta = 10 \cdot 2^{2n} + 2 \cdot 2^n.$$

From (14),

$$e = 25 \cdot 2^{2n} + 10 \cdot 2^n - 1.$$

Thus (c, d, e) forms a Diophantine triple with the property $D(8 \cdot 2^{2n})$.

Case IV

Let $d = 9 \cdot 2^{2n} + 6 \cdot 2^n - 1$ and $e = 25 \cdot 2^{2n} + 10 \cdot 2^n - 1.$

Let f be any non-zero integer.

Consider

$$df + 8 \cdot 2^{2n} = \alpha^2,$$

$$ef + 8 \cdot 2^{2n} = \lambda^2. \tag{15}$$

Using some algebra,

$$e\alpha^2 - d\lambda^2 = (e - d)(8 \cdot 2^{2n}).$$

Using the linear transformations,

$$\alpha = X + dT \quad \text{and} \quad \lambda = X + eT$$

and $T = 1$, we have

$$X = 15 \cdot 2^{2n} + 8 \cdot 2^n - 1 \quad \text{and} \quad \alpha = 24 \cdot 2^{2n} + 14 \cdot 2^n - 2.$$

From (15),

$$f = 64 \cdot 2^{2n} + 32 \cdot 2^n - 4.$$

Thus (d, e, f) forms a Diophantine triple with the property $D(8 \cdot 2^{2n})$.

Case V

Let $e = 25 \cdot 2^{2n} + 10 \cdot 2^n - 1$ and $f = 64 \cdot 2^{2n} + 32 \cdot 2^n - 4$.

Let g be any non-zero integer.

Consider

$$eg + 8 \cdot 2^{2n} = \phi^2,$$

$$fg + 8 \cdot 2^{2n} = \varphi^2. \tag{16}$$

Using some algebra,

$$f\phi^2 - e\varphi^2 (f - e) 8 \cdot 2^{2n}.$$

Using the linear transformations,

$$\phi = X = eT \quad \text{and} \quad \varphi = X = fT$$

and $T = 1$, we have

$$X = 40 \cdot 2^{2n} + 18 \cdot 2^n - 2 \quad \text{and} \quad \phi = 65 \cdot 2^{2n} + 28 \cdot 2^n - 3.$$

From (16),

$$g = 169 \cdot 2^{2n} + 78 \cdot 2^n - 9.$$

Thus (e, f, g) forms a Diophantine triple with the property $D(8 \cdot 2^{2n})$.

From all the above cases, $(a, b, c), (b, c, d), (c, d, e), (d, e, f), (e, f, g), \ldots$ will form a sequence of Diophantine triples $D(8 \cdot 2^{2n})$.

n	(a, b, c)	(b, c, d)	(c, d, e)	$D(8 \cdot 2^{2n})$
0	$(-2, 2, 4)$	$(2, 4, 14)$	$(4, 14, 34)$	$D(8)$
1	$(-1, 7, 16)$	$(7, 16, 47)$	$(16, 47, 119)$	$D(32)$
2	$(7, 23, 64)$	$(23, 64, 167)$	$(64, 167, 439)$	$D(128)$
3	$(47, 79, 256)$	$(79, 256, 623)$	$(256, 623, 1679)$	$D(512)$
4	$(223, 287, 1024)$	$(287, 1024, 2399)$	$(1024, 2399, 6559)$	$D(2048)$

3. Conclusion

To conclude one may construct a sequence of Diophantine triples with suitable properties.

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