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THE MONETARY SCALE AND THE SCALE OF UTILITY WITH RESPECT TO THE ADDITIVITY PROPERTY OF SUBJECTIVE PROBABILITY

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Abstract

Clarifying similarities and differences between the monetary scale and the scale of utility is carried out by means of the comparison between the assumption of rigidity in the face of risk and risk aversion. This comparison enables us to scrutinize the additivity property of the price of a random gain which underlies the operational definition of probability. Such a property of coherence is not a simplifying hypothesis which could be approximately valid, but it is an exact property which is the foundation of probability theory when we define the price of a random gain on the basis of the monetary scale instead of the scale of utility. This is because the notion of certain gain which is equivalent to a random gain is based on the scale of utility of a risk-neutral individual.

1. Introduction

The foundations of the theory of choice under conditions of uncertainty are probability and utility. They are equally, but also in different meaning, subjective notions. For the former of these notions, only after recognizing the essential

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characteristics of its more typical and objective elements, which are random events and random numbers, it is convenient to come to an operational definition on the basis of which outcomes of a decision-making are certainly unpleasant if one breaks the objective condition of coherence of probability. More explicitly, we mean that any transgression of this condition results in decisions whose consequences lead to certain loss (see [1], [2], [5] and [12]). For the latter, an interval scale is again introduced, since the cardinal utility concept is restored: if an individual prefers b to a and c to b, with a, b, c which are not uncertain preferences, by virtue of the ordinal utility notion introduced by Pareto, we can only say that a utility function u such that $u(a) < u(b) < u(c)$ exists, with the actual numbers which are meaningless. Conversely, the cardinal utility theory claims that differences between preferences are important: it is not only meaningful to ask which option is better than the other but it is meaningful to ask how much it is better (see [3], [6], [7] and [8]). Consequently, with X, Y and Z which are random numbers, if we consider $u(X) < u(Y) < u(Z)$, the actual numbers which take the place of $u(X)$, $u(Y)$ and $u(Z)$ tell us if $u(Y)$ is closer to $u(X)$ or to $u(Z)$ and how much it is closer.

2. Different Types of Uncertainty

Deciding means identifying one of different and possible actions. This identification can unconsciously occur because one does not even consider that many alternatives exist in addition to the one that one is going to put into practice or, even if it is possible to distinguish them, they are removed without a proper and rigorous analysis. Although it cannot be excluded with certainty that perspicacity and instinct of an individual can often lead to decide correctly, it is wrong to believe that one will always be able to arrive at an optimal decision through these faculties. In particular, when the problem to be faced is complicated, it is convenient studying well all the conditions and perspectives of the considered situation, representing first the set of possible outcomes and subsequently weighing carefully advantages and disadvantages of each of these outcomes. In general, since outcomes of an action are uncertain, they turn out to be foreseeable only in an inaccurate fashion. Therefore, when we study this, we express a judgment on such forecasts or, more in detail, on probabilities that are subjectively assigned to different and possible outcomes of each decision. Every decision is economic when considerations of convenience can be expressed in terms of costs, gains, wants and so on. It must be present, and in a not insignificant fashion, a cost which must be compared with other elements. So, stipulating or not an insurance is an economic decision, a decision concerning convenience of a certain company to making a certain investment is an economic decision, actions that a certain government takes in economic field are economic decisions or decisions of economic policy, a decision of a certain individual of joining or not a sports club on payment of a financial contribution, absolutely not insignificant, is an economic decision. Thus, if a debtor obtains the right to delay a payment to a creditor for a defined period of time, it is necessary to calculate the present value of this payment in order to consider the discounting. Probabilities and previsions are necessary for expressing judgments when there is uncertainty, while if an amount of money is considerable for a certain individual, it is necessary to replace the monetary scale with the one of utility which must consider a risk aversion more or less strong. It often happens to deciding without having certain knowledge of the conditions on which we would like to base our decisions and it often occurs that outcomes of decisions that could be taken are unknown. Practically, conditions of certainty never exist and they are mentioned only to facilitating problems when uncertainty, which is inherent in them, is neglected as irrelevant or it is entirely omitted for needs of initial simplification. Anyway, we can distinguish different types of uncertainty: ignorance, uncertainty proper, risk, competitive uncertainty. In the first case, uncertainty depends only on a lack of information. In the second, uncertainty concerns more or less unforeseeable events and if they occur, then a given activity may cease to be beneficial. In the third case, uncertainty concerns events whose consequences could be eliminated by entering into insurance. In the final case, uncertainty is related to the competition studied by game theory. Evidently, in all these cases in which a coherent decision must be taken choosing it into a set of possible options, a coherent criterion of decision-making primarily requires an evaluation of probability. Secondly, if we regard considerable problems with respect to the means of an individual who must decide is also established a scale of utility (see [3], [7], [8] and [9]).

3. Random Events and Random Numbers

Probability is a subjective notion that should be separated from identification of the set of possible alternatives on which objectively extends our uncertainty.

Studying the domain of possible alternatives means identifying everything that can be said about uncertainty remaining in the domain of what is objective. In this domain, one simply realizes what one knows with certainty as well as what one does not know with certainty. Anyway, the most important elements to which subjective probability refers are random events and random numbers: they are objective elements (see [2] and [12]). An event E is a statement which we do not know yet to be true or false, while the event which is certain or true with certainty and the one which is impossible or false with certainty can be taken as a limit case. Statements of which we can say if they are true or false on the basis of an ascertainment well determined and always possible, at least conceptually, have objective meaning. Such objective statements are said *propositions* if one is thinking more in terms of the expressions in which they are formulated or, equally, events if one is thinking more in terms of the situations and circumstances to which their being true or false corresponds (see [4]). Thus, proposition and event are the same thing although such words are different. Evidently, the proposition “it will snow in Rome on 20 January 2018” is a random event, while “draw between Juventus and Torino in the match played on 5 December 1976” is a random event only for those who do not know or do not remember the result of the match, otherwise it is an event. For an individual, an event is certain or impossible when, before knowing if it is true or false, he already knows the outcome: the proposition “in a throw of a die having six faces, with each of them showing a different number from 1 to 6, the face that is uppermost when it comes to rest is 1 or 2 or 3 or 4 or 5 or 6” is an event but it is not a random event as well as “in a throw of a die having six faces, with each of them showing a different number from 1 to 6, the face that is uppermost when it comes to rest is 41”. Clearly, the intermediate case of uncertainty is the only that allows evaluations of probability (see [10]).

For any individual who does not know with certainty the numerical value of a number X, which is random in a non-redundant usage for him, there are two or more than two, a finite or infinite number, possible values for X. The set of these values is denoted by $I(X)$: in any case, only one is the true value of each random number and the meaning that we have to give to the adjective “random” is the one of not known by the individual of whom we consider his state of uncertainty. Thus, “random” does not mean undetermined but, on the contrary, it means established unequivocally, so a supposed bet based upon it would be indisputably decided at the appropriate time. The possible values for X are objective because, although it is personal their determination, they depend on objective circumstances which consist in the imperfect state of information of an individual, that is to say, in his degree of ignorance. In fact, when a given individual outlines the domain of uncertainty he does not use his subjective opinions on what he does not know because the values of X depend only on what he objectively knows or not. Every random number has a probability distribution which is an expression of the attitude of an individual. Hence, such a distribution can change from individual to individual depending on information of each of them.

Fields in which we can have uncertainty are different: it can concern not only economic events but also political events, meteorological phenomena, scientific conjectures, judicial investigation, competition in sport, personal or everyday affairs and so on. Evidently, interpretation which must be given to the domain of uncertainty is absolutely not reductive since what is uncertain or possible does not concern only the restricted field of games of chance (see [2] and [3]).

Remark 1. Events are questions whose wordings, unambiguous and exhaustive have the aim of removing any opportunity to complain in case that a bet is based upon them: they admit two alternative answers, $\text{yes} = 1$ or $\text{no} = 0$, $\text{true} = 1$ or $\text{false} = 0$. With regard to the previous fields in which we can have uncertainty, we can ask ourselves: will the thirteenth President of the Italian Republic be a woman? In 2017, will the Italian gross domestic product increase by 1% compared to last year? Will it snow in Rome on 20 January 2018? Will man reach the planet Mars by 2016? Will Mario Bianchi, accused of being the murderer of his wife, be sentenced in the criminal trial of first instance in which he is accused? In the season 2016-2017 will the Italian football championship be won by Juventus? Will the university student Fabio Rossi pass his chemistry examination? Will the bus I am waiting for come at midday? In a throw of a die having six faces, with each of them showing a different number from 1 to 6, will the face that is uppermost when it comes to rest be 6? (see [1] and [2]).

Remark 2. Also random numbers can be identified by questions whose wordings are indisputably clear and complete; unlike events, they contain two or more than two answers which consist only of numbers, only one of which is the one that actually occurs. More in detail, we can ask ourselves: how many will the votes in favour for the election of the President of the Italian Republic be? Or, in the voting that will elect a new President, how many will blank ballot papers be? Or, how many will spoiled ballot papers be? Or, what will the percentage of votes in favour be? In 2017 what will the value, expressed in US dollars, of the Italian gross domestic product be? In which instant will man reach Mars? On which day will the judge of first instance decide about accusation addressed to Mario Bianchi? What will the mark obtained by Fabio Rossi be? What will the number of points obtained by Juventus in the final league table of the Italian football championship 2016-2017 be? (see [2]).

4. Operational Definition of Probability

Subjective probability is the degree of belief of a certain individual for a certain event which could occur. Operationally, from the concept of a fair bet is deduced the numerical measure of subjective probability: a bet which is based upon an event E is fair, for a given individual, when it may be accepted in both ways, that is to say, he could be either better or bookmaker. Thus, supposing that he is a better, if he believes fair exchanging, for any amount S positive or negative, an amount pS which is certain (because it is certainly paid by the better) for the right of S on condition that E occurs, then the coefficient p is said *probability* of E and we have $p = \mathbf{P}(E)$ (with $0 \leq \mathbf{P}(E) \leq 1$). If $S = 1$, p is the price to pay to receiving, when E occurs, a unit amount. Clearly, if E does not occur, a better loses p. On the other hand, he loses pS in case that we have $S \neq 1$ (see [1]).

Remark 3. For a better, the gain of a bet upon a possible event E is given by $G = (E - p)S$, with $p = \mathbf{P}(E)$ and $S \neq 0$ arbitrary amount. This gain, by virtue of the fact that it turns out to be $G = (1 - p)S + (0 - p)S$, is always a positive or negative random number: if we exclude $p = 0$ and $p = 1$ for $\mathbf{P}(E)$ and if we suppose $S > 0$, we have $G_1 = (1 - p)S > 0$ in case that E occurs, otherwise it will be $G_2 = -pS < 0$. Then, every bet which is based upon E is coherent if and only if we have $0 < p < 1$, with p amount which is paid by the better when we have $S = 1$. Hence, it follows that G_1 and G_2 are not of the same sign, that is to say, $G_1G_2 < 0$. Nevertheless, in consideration of $p = \mathbf{P}(E) = 0$ or $p = \mathbf{P}(E) = 1$ for a possible event E, every bet upon E is coherent if and only if we have $0 \leq p \leq 1$, with p amount which is paid by the better in case that $S = 1$. More exhaustively, it will be

$G_1 G_2 \leq 0.$ Moreover, we must have $p = 0$ in order for a bet upon an impossible event $E = \varnothing$ is coherent and $p = 1$ in order for a bet which is based upon a certain event $E = \Omega$ is coherent: in such cases we have $G = G_2 = 0$ and $G = G_1 = 0.$ Evidently, given $S > 0,$ when better becomes bookmaker and vice versa, G_1 and G_2 are subjected to a change of sign (see [5] and [12]).

Remark 4. Let ε be an algebra of events which are finite in number and let $\{E_1, \ldots, E_n\}$ be a subclass of incompatible and exhaustive events of ε such that we have $E_i \wedge E_k = \varnothing$ $(i \neq k),$ $\vee_{i=1}^{n} E_i = \Omega,$ with $\wedge$ logical product and $\vee$ logical sum: if we consider a combination of bets each of which is based upon one of the random events of $\{E_1, \ldots, E_n\},$ with $p_i S_i$ amount of money that a better pays to receiving S_i when E_i occurs $(i = 1, \ldots, n),$ then the gain of such a combination is given by $G = \sum_{i=1}^{n} (E_i - p_i) S_i,$ where we have $p_i = p_i \, S_i / S_i = \mathbf{P}(E_i).$ Since the set $\{E_1, \ldots, E_n\}$ is a partition of $\Omega,$ betting upon each of the events of $\{E_1, \ldots, E_n\}$ is equivalent to betting upon $\Omega,$ with $\mathbf{P}(\Omega) = 1,$ because it is certain that one and only one event occurs. If we choose $S_1 = \cdots = S_n = S = 1,$ then it follows $G = 1 - (p_1 + \cdots + p_n).$ This combination of bets is coherent if and only if we have $G = 0.$ Evidently, by virtue of finite additivity of probability, it turns out to be $p_1 + \cdots + p_n = 1$ (see [12]).

Remark 5. Let ε be an algebra of events which are finite in number. If $\{E_1, \ldots, E_n\}$ is a subclass of ε whose events are mutually exclusive but they do not constitute a partition of a certain event $\Omega,$ then the probability of the logical sum of n (with n integer ≥ 2) mutually exclusive events is the sum of the probabilities of the single events, that is to say, we have $\mathbf{P}(E_1 \vee \cdots \vee E_n) = \mathbf{P}(E_1) + \cdots + \mathbf{P}(E_n)$ by virtue of finite additivity of probability. This property, which is expressed by the law of total probability, confirms that, according to the subjectivistic conception of probability, the theorems of the calculus of probability do not express any special qualities that must be satisfied in the real world but they are necessary and sufficient conditions in order for the opinions of a certain individual are not incoherent (see [5]). So, given that $(E_1 \vee \cdots \vee E_n)$ occurs if and only if one of the events of

$\{E_1, \ldots, E_n\}$ occurs, the left-hand side of the previous equality must be interpreted as the amount that a given individual believes fair to pay in order to receiving a unit amount when one and only one of the events of $\{E_1, \ldots, E_n\}$ occurs, while its right-hand side must be interpreted as the amount that the same individual believes fair to pay to receiving a unit amount only once. With the events of $\{E_1, \ldots, E_n\}$ which are mutually exclusive, logical and arithmetic sum coincide and we have $\mathbf{P}(E_1 \vee \cdots \vee E_n) = \mathbf{P}(E_1 + \ldots + E_n)$.

Remark 6. If $\{E_1, \ldots, E_n\}$ is an any subclass of ε, coherence requires to evaluating the probability of $(E_1 \vee \cdots \vee E_n)$ and the probabilities of the single events so that the inequality given by $\mathbf{P}(E_1 \vee \cdots \vee E_n) \leq \mathbf{P}(E_1) + \cdots + \mathbf{P}(E_n)$ is satisfied. The left-hand side of this inequality can be interpreted as the amount that a given individual believes fair to pay to receiving a unit amount if at least one of these events occurs, while the right-hand side is the amount that the same individual believes fair to pay to receiving as many unit amounts as true events appear (see [12]).

Remark 7. In case that X is a random number, $\mathbf{P}(X)$ is its prevision: if $I(X) = \{x_1, \ldots, x_n\}$, when we assign to each possible value x_i of X the probability p_i $(i = 1, \ldots, n)$, with $0 \leq p_i \leq 1$ and $\sum p_i = 1$, it turns out to be $\mathbf{P}(X) = x_1 p_1 + \cdots + x_n p_n$. The prevision of X coincides with the probability of an event E when and only when X, admitting only two possible values, 1 and 0, is an event, that is, $X = E$. Thus, prevision and probability are two different words that express the same concept extra-logical, subjective and personal (see [2]).

Remark 8. If the sets of possible values for X and Y are, respectively, $I(X) = \{x_1, \ldots, x_n\}$ and $I(Y) = \{y_1, \ldots, y_n\}$, when we assign the same weights p_i $(i = 1, \ldots, n)$ to each x_i and y_i, where we have $0 \leq p_i \leq 1$ and $\sum p_i = 1$, we will have $\mathbf{P}(X + Y) = \mathbf{P}(X) + \mathbf{P}(Y)$, that is to say, $\mathbf{P}$ is additive. A prevision $\mathbf{P}$ of the random number X must satisfy the inequality $\inf I(X) \leq \mathbf{P}(X) \leq \sup I(X)$, that is, $\mathbf{P}(X)$ must not be less than the lower bound of the set of possible values for X, which is $\inf I(X)$, nor greater than the upper bound, which is $\sup I(X)$. A prevision

$\mathbf{P}$ of X must also be linear, that is, we have $\mathbf{P}(aX) = a\mathbf{P}(X)$, for every real number a. In general, we have to consider $\mathbf{P}(aX + bY + cZ + \cdots) = a\mathbf{P}(X) + b\mathbf{P}(Y) + c\mathbf{P}(Z) + \cdots$, with $a, b, c, \ldots$ whatever real numbers, for any finite number of summands. Similarly, if E is an event, when we have $0 \leq \mathbf{P}(E) \leq 1$, its evaluation of probability is coherent; if $E_1, \ldots, E_n$ are mutually exclusive events, their evaluations are coherent when we have $\mathbf{P}(E_1 + \cdots + E_n) = \mathbf{P}(E_1) + \cdots + \mathbf{P}(E_n)$. Thus, for every prevision of one or more random numbers, the condition of coherence reduces to linearity, which contains additivity property, and convexity. In particular, it reduces to not negativity of any evaluation of probability of a random event and finite additivity (see [2] and [12]).

5. Utility Function of a Risk-averse Individual

For a risk-averse individual, the increments of the scale of utility $u(x)$ are equal, on the vertical axis of a two-dimensional Cartesian coordinate system, when and only when the corresponding indifferent increments of the variable x, on the horizontal axis, are greater and greater. Hence, $u(x)$, which is a continuous and strictly increasing function, is a concave function. In other words, successive and equal values of the variable x on the horizontal axis have less utility on the vertical axis. Anyway, it would be more appropriate that the variable x, on the horizontal axis, does not represent the current monetary gain of an individual but it represents his means expressed as an algebraic sum of his gain and his fortune. With regard to his fortune, one could think of the value of his estate. It is convenient to taking into account such a sum as a less arbitrary origin because judgments of an individual might alter when variations in one's fortune have occurred or risks have been taken (see [2], [3] and [8]).

Remark 9. In order to base judgments of indifference of a risk-averse individual on a given scale of utility $u(x)$, we have to consider every indifferent increment of the variable x in an appropriate interval on the x-axis. If we have $u(x) = \ln x$ $(x > 0)$ and if we consider, on the x-axis, the interval from 0 to 32, we can subdivide it into two parts to which correspond two subintervals, the subinterval from 0 to $S = 16$ and the subinterval from $S = 16$ to $2S = 32$. Then, $S' = 12, 8$ and

$2S = 32$ are the two monetary values on the x-axis for which $(32 - 12, 8)$ is an indifferent increment, while $(\ln 32 - 12, 8 = \ln 2, 5)$ is the corresponding increment of utility on the vertical axis. If we consider the interval from 0 to $64,$ we can subdivide it into four parts to which correspond four subintervals, the subinterval from 0 to $S = 16,$ the subinterval from $S = 16$ to $2S = 32,$ the subinterval from $2S = 32$ to $3S = 48$ and the one from $3S = 48$ to $4S = 64.$ Then, $2S' = 25, 6$ and $4S = 64$ are the two monetary values on the x-axis for which $(64 - 25, 6)$ is an indifferent increment, while $(\ln 64 - \ln 25, 6 = \ln 2, 5)$ is the same increment of utility on the y-axis. In the interval from 0 to $128,$ $4S' = 51, 2$ and $8S = 128$ are the two monetary values on the x-axis for which $(128 - 51, 2)$ is an indifferent increment, while $(\ln 128 - \ln 51, 2 = \ln 2, 5)$ is the same increment of utility on the y-axis. Conversely, in the interval from 0 to $256,$ $8S' = 102, 4$ and $16S = 256$ are the two monetary values on the x-axis for which $(256 - 102, 4)$ is an indifferent increment, while $(\ln 256 - \ln 102, 4 = \ln 2, 5)$ is the same increment of utility on the y-axis. Clearly, in order to construct a scale of utility we can consider greater and greater intervals on the x-axis. Anyway, additivity property is valid also for the monetary values of the variable x of the function $u(x)$ by virtue of the fact that we have $2S = S + S,$ $4S = S + S + S + S$ and so on (see [2]).

Remark 10. A function $u(x)$ of a risk-averse individual is more or less concave according to his more or less prominent aversion on which it is possible that various circumstances, including the current mood, have influence. The scale of utility $u(x)$ is unique up to an affine transformation so that any function $au(x) + b,$ with a which is a positive constant and b which is an arbitrary constant, is equivalent to $u(x)$ in order to representing individual preferences. In other words, changes of origin and unit of measurement are inessential. If $b = 0,$ $au(x)$ is a linear transformation of $u(x)$ and $au(x)$ is a strictly monotonic function like $au(x) + b$ (see [7] and [9]).

Remark 11. Let $u(x)$ be the utility of the monetary gain $x.$ According to the theory of decision-making, the increments of utility from $u(x_{i-1})$ to $u(x_i)$ and from $u(x_i)$ to $u(x_{i+1})$ are always equal, for a risk-averse individual, when and only when it is indifferent for him the choice between the possession of the first increment x_i of monetary gain and a lottery whose possible values are x_{i-1} and x_{i+1} with equal probability, $1/2$ and $1/2$ (see [8]). In other words, there are two and equally advantageous alternatives for him: keeping x_i or playing heads or tails losing x_i if it is tails, "doubling" x_i if it is heads, it being understood that losing x_i coincides with "obtaining" x_{i-1}, "doubling" x_i coincides with obtaining x_{i+1} and, because of risk aversion, x_i is not closer to x_{i+1} but it is closer to x_{i-1}.

Remark 12. We can measure the utility when we extend the preference order to situations not only certain but also uncertain. So, after fixing x_0 and x_1 on the x-axis to which correspond $u(x_0) = 0$ and $u(x_1) = 1$ on the y-axis, $x_{1/2}$ is included between x_0 and x_1. Such a value, to which corresponds $u(x_{1/2}) = 1/2$ on the y-axis, is equivalent to the random possession of x_0 or x_1 with equal probability, $1/2$ and $1/2$. The interval from x_0 to $x_{1/2}$ can further be divided considering the point $x_{1/4}$ to which corresponds $u(x_{1/4}) = 1/4$ and the same thing can be made considering the point $x_{3/4}$ of the interval from $x_{1/2}$ to x_1 to which corresponds $u(x_{3/4}) = 3/4$ on the vertical axis. Similarly, x_1 is equivalent to the random possession of x_0 or x_2 with equal probability, $1/2$ and $1/2$. The point x_2 on the horizontal axis corresponds to $u(x_2) = 2$ on the vertical axis (see [3]).

Remark 13. When the probability of a random event E is $1/2$ for a given individual, it is indifferent for him accepting a bet which is based upon E or $\overline{E}$, with $\overline{E}$ which is the negation of E. Conversely, if it is indifferent for a certain individual betting upon E or $\overline{E}$, then the probability of E is $1/2$ for him (see [6]).

6. Certain Gain which is Equivalent to a Random Gain

A random gain X is a random quantity having meaning of monetary gain. This gain must be intended in an algebraic meaning for which the possible values of X could be not only all positive but also, entirely or only in part, negative, confirming the fact that a loss is a negative gain.

Remark 14. If X is a random gain whose possible values are $x_1, \ldots, x_n$ for a given individual, then the price of X is $\mathbf{P}(X)$ according to his state of information and his opinions. Such a price coincides with the certain gain which is equivalent to X. If $I(X) = \{x_1, \ldots, x_n\}$, when we assign to each numerical value x_i of X the probability p_i $(i = 1, \ldots, n)$, with $0 \leq p_i \leq 1$ and $\sum p_i = 1$, it turns out to be $\mathbf{P}(X) = x_1 p_1 + \cdots + x_n p_n$ for the price of X. In order to deciding coherently under conditions of uncertainty, it is necessary to inserting the degree of preferability of a random gain into the subjective scale $u(x)$ of the certain gains. Afterwards, X is preferred to x, where x is the certain gain, when we have $x > \mathbf{P}(X)$, while x is preferred to X when it turns out to be $x < \mathbf{P}(X)$. Conversely, the gains x and X are indifferent when it results $x = \mathbf{P}(X)$. In general, any price always measures a preference which must manifest itself in one way or another. Therefore, if we have $\mathbf{P}(X)$ and $\mathbf{P}(Y)$, with $\mathbf{P}(X) \neq \mathbf{P}(Y)$, X is preferred to Y or Y is preferred to X according to which price is highest of the other (see [2] and [11]).

Remark 15. A risk-averse individual always prefers the certain alternative to the uncertain one: he always prefers x to X. Since his utility function $u(x)$ is a concave function, with regard to associative means, we have $x < \mathbf{P}(X)$ on the horizontal and positive half-axis. We consider such a half-axis only for the sake of convenience. It is absolutely normal that the barycentre determined by the point masses, which are placed on a curve of the Cartesian plane, is not, in general, a point of the curve. Thus, if we place the weights or masses p_h $(h = 1, \ldots, n)$ on a portion of the curve with concavity downwards, the barycentre is always in the area bounded by such a concavity because it is a point of the convex hull which is generated by all the convex combinations of points $(x_h, u(x_h))$ when the real coefficients p_h vary in the interval $[0, 1]$. Analytically, since $u(x)$ is an invertible function, it turns out to be $x = u^{-1}(u(x))$, that is to say, x is an associative mean which coincides with the abscissa of a point on the graph of $u(x)$, unlike $\mathbf{P}(X)$ which is the weighted arithmetic mean of the possible values x_i $(i = 1, \ldots, n)$ for X. Symmetrically, if an individual is said to be *risk-loving*, his utility function $u(x)$, where we have $x > 0$, is a convex function and it results $x > \mathbf{P}(X)$ again by virtue of the comparison between associative means on the horizontal and positive half-axis. On the contrary, if an individual is said to be *risk-neutral*, his utility function $u(x)$, with $x > 0$, is a linear function and it turns out to be $x = \mathbf{P}(X)$ on the horizontal and positive half-axis (see [2]).

Remark 16. In general, because of risk aversion, it is not true that if a given individual is prepared to buy an article A at the price $\mathbf{P}(A)$ and an article B at the price $\mathbf{P}(B)$, he must be prepared to buy both of them together at the price $\mathbf{P}(A) + \mathbf{P}(B)$. In effect, it could happen that the purchase of one of them affects, in different ways, the desirability of the other and the same thing is valid if we consider two random gains X and Y instead of two articles A and B. If we admit the additivity hypothesis, then we are prepared to buy both articles A and B at the price $\mathbf{P}(A) + \mathbf{P}(B)$ or both random gains X and Y at the price $\mathbf{P}(X) + \mathbf{P}(Y)$. Such a simplifying hypothesis means that if a given individual is indifferent to the exchange of X for $\mathbf{P}(X)$ and of Y for $\mathbf{P}(Y)$, then he is indifferent to the exchange of $X + Y$ for $\mathbf{P}(X) + \mathbf{P}(Y)$. Since $\mathbf{P}$ expresses a subjective evaluation, the value for which he is indifferent to the exchange of $X + Y$ is, by definition, $\mathbf{P}(X + Y)$. Hence, we have $\mathbf{P}(X + Y) = \mathbf{P}(X) + \mathbf{P}(Y)$ according to additivity property of $\mathbf{P}$ (see [2]).

Remark 17. Accepting additivity property of $\mathbf{P}$ means that we do not face risk aversion but we face risk neutrality. Thus, $\mathbf{P}(X)$ and X are indifferent for a risk-neutral individual and they can be exchanged. It follows that when we simultaneously and in parallel establish the properties of probability and utility we are constructing the theory of decision-making in an integrated fashion whose meaning is unexceptionable from an economic viewpoint. However, when the notion of utility is set aside because it is not necessary, additivity property is connected to the monetary scale instead of the scale of utility, with such a connection which must occur within appropriate limits. Clearly, this approach is not any more a unified approach to an integrated formulation of decision theory in its two components, probability and utility (see [2]).

Remark 18. In general, if X is not a random gain but it is a random quantity whose possible values are not monetary values, then $\mathbf{P}(X)$ is not the price of X but it is its prevision or, more in particular, when $X = E$ is a random event, $\mathbf{P}(E)$ is the probability of E (see [2]).

7. Comparison between Rigidity with respect to Risk and Risk Aversion

The prevision $\mathbf{P}$ of a random number and the probability $\mathbf{P}$ of a random event are based on the additivity property of the price $\mathbf{P}$ of a random gain. Such a property expresses an assumption of rigidity in the face of risk which is opposed to the hypothesis of risk aversion. Indeed, judgments of indifference of an individual, who is rigid in the face of risk, are based on the monetary scale which is objective, while they are based on the subjective scale of utility when we face risk aversion.

Remark 19. A and B are two incompatible and exhaustive random events, that is to say, it is certain that one and only one of them occurs. If A and B are equally probable for a given individual, then it results $\mathbf{P}(A) = \mathbf{P}(B) = 1/2$ because from $A + B = 1$ it follows that it turns out to be $\mathbf{P}(A) + \mathbf{P}(B) = \mathbf{P}(A + B) = \mathbf{P}(1) = 1$ according to the additivity property of probability. Supposing that a given individual is faced with two possibilities which he judges equally probable, according to the standard example which is related to this hypothesis we consider a question of heads or tails. Therefore, we have $A =$ "it is heads in a coin toss", $B =$ "it is tails in a coin toss".

Remark 20. According to the hypothesis of rigidity in the face of risk, for an individual who believes that two incompatible and exhaustive eventualities are equally probable, it is indifferent receiving with certainty a sum S or twice this sum if one and only one of the two possible cases occurs. Thus, if he bets upon the fact that it is heads in a coin toss, since the random gain $2S$ is conditional on the random event A, $2SA$ is the random number for which we have $\mathbf{P}(2SA) = 2S\mathbf{P}(A) = 2S\,1/2 = S$. Evidently, such a monetary value coincides with the certain sum S. Similarly, for an individual who is rigid in the face of risk, it is indifferent losing with certainty a sum $-S$ or twice this sum if one and only one of the two possible cases occurs. Hence, if he bets upon the fact that it is heads in a coin toss, since the gain $-2S$ is conditional on the random event A, $-2SA$ is the random quantity for which it turns out to be $\mathbf{P}(-2SA) = -2S\mathbf{P}(A) = -2S\,1/2 = -S$. Clearly, such a monetary value coincides with the certain sum $-S$. Moreover, for the same individual, it is indifferent accepting or not accepting a bet which would lead him to a gain S or to a loss $-S$. In effect, if he does not bet, he cannot win as well as he cannot lose, therefore, the certain sum that belongs to him is 0. Conversely, if he bets, he wins SA when the random event A occurs, while he loses $-SB$ when the random event B occurs, consequently, we have $\mathbf{P}(SA) + \mathbf{P}(-SB) = S/2 - S/2 = 0$. Obviously, the monetary values coincide because we get the number 0 in the two possible cases (see [2]).

Remark 21. The certain alternative and the uncertain one are indifferent, for an individual who is rigid in the face of risk, when and only when the monetary values corresponding to them are the same. Evidently, such values are objective.

Remark 22. In all cases a risk-averse individual prefers the certain alternative to the uncertain one. Thus, he would content himself with receiving with certainty a sum S' which is less than S in exchange for the hypothetical gain $2S$. Likewise, he would be disposed to pay with certainty a sum $-S''$ whose absolute value is greater than $-S$ in order to avoid the risk of a hypothetical loss $-2S$. Similarly, he would pay with certainty a penalty $-K$ in order to be released from any bet in which he would lose $-S$ or he would win S, with such monetary values which are symmetric. With regard to the scale of utility $u(x)$, we have equal levels on the vertical axis in passing from 0 to S' and from S' to $2S$ on the horizontal and non-negative half-axis or in passing from $-2S$ to $-S''$ and from $-S''$ to 0 on the horizontal and non-positive half-axis or in passing from $-S$ to $-K$ and from $-K$ to S on the horizontal axis. A lottery whose possible values are 0 and $2S$ is equal to S because such a value coincides with the weighted arithmetic mean of 0 and $2S$ whose weights are $1/2$ and $1/2$. Clearly, S is the expected value of the lottery. We consider $S' < S$ by virtue of risk aversion, with S' which is closer to 0 than $2S$. In the other cases, we have $-S'' < -S$ and $-K < 0$. Evidently, this conforms to the properties of associative means (see [2]).

8. Criteria of Coherent Decisions under Conditions of Uncertainty

Probability theory is based on the study of decisions under conditions of uncertainty. For this reason, it is necessary considering the criterion consisting of testing, through the decisions of an individual which are observable, his subjective opinions, that is to say, his previsions or probabilities which are not directly observable. Afterwards, it needs to verify if such opinions are or are not coherent and they must be changed if they are not coherent. The comparison between the monetary scale and the scale of utility enables us to scrutinize the additivity property of the price of a random gain which underlies the operational definition of probability. Such a property of coherence is not a simplifying hypothesis which could be approximately valid, but it is an exact property which is the foundation of probability theory when we define the price of a random gain on the basis of the monetary scale instead of the scale of utility. This is because the notion of certain gain which is equivalent to a random gain is based on the scale of utility of a risk-neutral individual. In general, when a given individual chooses one or more probabilities or a utility function according to his subjective preferences, he must consciously choose. Obviously, if he can free himself from laziness and if he can protect himself from whim, then he consciously chooses. Anyway, both the direct interest and the lack of it could psychologically on the one hand encourage and on the other obstruct the calmness and accuracy and, consequently, the reliability of these choices. Moreover, reliable choices can be made when an individual is consulted about choices in which others are interested. On the other hand, we consider a situation which is similar to the last one when calmness and accuracy are related to one's self-respect in some competitive situation whose prizes are connected to the significance of the competition, although they are materially insignificant.

Remark 23. One of the two fundamental components of the theory of choice is utility function $u(x)$. To the certain gain x on the horizontal axis which is equivalent to the random gain X corresponds the function $u(X) = u(x_1)\,p_1 + \cdots + u(x_n)\,p_n$ on the vertical axis. For this function, the possible values x_i of X and their probabilities p_i, where we have $0 \leq p_i \leq 1$ and $\sum p_i = 1$, are the same of $\mathbf{P}(X)$. The function $u(X)$ is the prevision of the utility and it is the weighted arithmetic mean of $u(x_i)$. The coherent criterion of decision-making for an individual consists in the subjective choice of any evaluation of the probabilities and any utility function according to his preferences and in fixing as one's goal the maximization of the prevision of the utility. Thus, the monetary gain X is preferred to the monetary gain Y if and only if we have $u(X) > u(Y)$. When a utility function is linear we observe that among the decisions which lead to different random gains, the choice must be the one which leads to the random gain with the highest price. Indeed, given X, the certain gain which is equivalent to X always coincides with the price $\mathbf{P}(X)$ on the horizontal axis to which corresponds $u(X)$ on the vertical axis (see [6], [7], [8] and [9]).

Remark 24. The contrast between the assumption of rigidity in the face of risk and risk aversion does not occur when the monetary amounts in consideration are not too large with regard to the means of a given individual. Hence, we can set aside the notion of utility, also when we are interested in applications of an economic nature with respect to which such a notion is properly used. This is because the coherent criterion of decision-making for an individual is not the one that leads to the maximization of the prevision of the utility but it must be the one that leads to the maximization of the prevision of the random gain (see [6] and [8]).

9. Reasons of the Separation of Probability from Utility

For different reasons it is appropriate to separate probability from utility, it being understood that probability is independent of risk aversion while utility is not independent of risk aversion. This separation enables us to accepting the hypothesis of rigidity in the face of risk with an important restriction. Indeed, it must be accepted within the limits that we can call "everyday affairs". With regard to the scale of utility $u(x)$ $(x > 0)$ of a risk-averse individual, successive increments of equal monetary value on the horizontal half-axis have for him smaller and smaller subjective utility on the vertical axis. Hence, $u(x)$ is a concave function. Conversely, for an individual who is rigid in the face of risk, the monetary value on the horizontal half-axis coincides with his utility on the vertical axis. Analytically, this qualification is satisfied by the equation of the line that bisects the first quadrant of a two-dimensional Cartesian coordinate system. Then, the above-mentioned restriction is relative and approximate in connection with $u(x)$. It is relative because it depends on the degree of concavity of $u(x)$ and it is approximate because we substitute in place of the portion of $u(x)$ which is of interest the tangent line at the starting point. Evidently, the smaller the interval considered, the more acceptable is the approximation.

Remark 25. We denote by $\mathbf{P}_u(X)$ the price of *X* for a given individual. This is because its original definition is based on the scale of utility $u(x)$ of a risk-neutral individual which is analytically a linear function. Thus, we could replace such a definition with a new one by means of the relation $\mathbf{P}(X) = \lim_{a \to 0}\left(\frac{1}{a}\right)\mathbf{P}_u(aX)$ in which, if a becomes too small, an evaluation $\mathbf{P}_u$ really loses any reliability. This form of the passage to the limit effectively separates the two scales but it is not appropriate to fulfill the function of a definition. Therefore, we prefer not changing anything, it being understood that in economic examples one must remain within appropriate monetary limits with respect to one's means (see [2]).

Remark 26. If $\mathbf{P}(X)$ is the linear prevision of the random gain X and if the degree of concavity of $u(x)$ is known, then it results $\mathbf{P}_u(X) = u^{-1}\{\mathbf{P}[u(X)]\}$, that is to say, $\mathbf{P}_u$ is expressed as a transform of $\mathbf{P}$ by means of u. Clearly, $\mathbf{P}(X)$ requires the monetary scale unlike $\mathbf{P}_u(X)$ which requires the scale of utility. With respect to the properties of associative means it turns out to be $\mathbf{P}_u(X) < \mathbf{P}(X)$, with $\mathbf{P}_u$ and $\mathbf{P}$ which are different notions. In fact, $\mathbf{P}_u(X)$ is the abscissa of a point on the graph of $u(x)$ unlike $\mathbf{P}(X)$ which is the abscissa of a point that is not on the graph of $u(x)$ because it is the barycentre of n (with n integer ≥ 1) point masses on $u(x)$ (see [2]).

Remark 27. With respect to the two different concepts $\mathbf{P}_u$ and $\mathbf{P}$, we can say that a transaction is indifferent when $\mathbf{P}_u$ remains constant, while it is fair when $\mathbf{P}$ remains constant. In general, any financial transaction under conditions of uncertainty consists of exchanging X_1 for X_2, where we have that X_1 is the random number before this exchange, X_2 is the random number after this exchange and $G = X_2 - X_1$ is the random gain with regard to this exchange. More in detail, a transaction is indifferent for a given individual when it turns out to be $\mathbf{P}_u[u(X_2)] - \mathbf{P}_u[u(X_1)] = 0$, it is advantageous when we have $\mathbf{P}_u[u(X_2)] - \mathbf{P}_u[u(X_1)] > 0$, while it is disadvantageous when it results $\mathbf{P}_u[u(X_2)] - \mathbf{P}_u[u(X_1)] < 0$. Conversely, a transaction is fair for a given individual when we have $\mathbf{P}(G) = 0$, it is favourable when we have $\mathbf{P}(G) > 0$, while it is unfavourable when it results $\mathbf{P}(G) < 0$ (see [2]).

Remark 28. The other reasons for preferring the separation of probability from utility concern the simplicity of this approach in comparison with the unified one. The main motivation concerns the monetary value in which combinations of bets and any other economic transactions are expressed. Such a value is objective or invariant unlike the scale of utility of a given individual which depends on his means and temperament and current mood. In any case, a utility function could also be influenced by some other circumstance. Using the prices **P** as they manifest themselves in the hypothesis of rigidity is similar to what is done in economics when one considers the total price $\mathbf{P}(X) = x_1 p_1 + \cdots + x_n p_n$ of a set of products, of given amounts x_i, on the basis of the unit prices p_i $(i = 1, \ldots, n)$ in force at the time, without taking into account the variation of each price that a possible transaction would cause by shifting the supply curve of each product or the demand curve or both of them. Anyway, these variations are noticeable if and only if the quantities under consideration are sufficiently large. The rigid approximation turns out to be acceptable also in actuarial mathematics where one often studies an insurance under fair conditions and only in special cases one speaks in terms of utility (see [2]).

10. Conclusions

A given individual coherently behaves with respect to his decisions and preferences when he fixes as one's goal the maximization of the prevision of the utility. Hence, he conforms one's way of thinking and acting to an evaluation of probability and a scale of utility which are underlying. This is the correct criterion of coherent decision-making which is based on a unified approach to an integrated formulation of decision theory in its two components. However, after defining **P** by means of a utility function of a risk-neutral individual, the notion of cardinal utility is set aside until a scale of utility is expressly required. Then, in accordance with the hypothesis of rigidity in the face of risk we use the monetary scale in order to justify the correctness of additivity property of **P**. Such a property of coherence is the foundation of probability theory.

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