**JP Journal of Mathematical Sciences**
Volume 22, Issue 1, 2018, Pages 1-16
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# TORSIONLESS MODULES OVER CLUSTER-TILTED ALGEBRAS OF TYPE AFFINE $A_n$

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## Abstract

In this paper, we study the properties of torsionless modules over cluster-tilted algebras of type affine $A_n$. All the indecomposable torsionless modules are local and they are uniserial modules when they are not projective. The number of all indecomposable nonisomorphism torsionless modules is only related to the number of arrows and oriented 3-cycles in the ordinary quivers and their relation expression is given.

## 1. Introduction

In 2001, Fomin and Zelevinsky [10, 11] introduced the concept of cluster algebras which rapidly become a successful research area. Cluster algebras nowadays link various areas of mathematics, like combinatorics, Lie theory, algebraic geometry, representation theory and also string theory in physics (via recent work on quivers with superpotentials [9, 12]).

In an attempt to "categorify" cluster algebras, cluster categories have been

2010 Mathematics Subject Classification: 18G25.

Keywords and phrases: torsionless modules, cluster-tilted algebras, quivers.

The author is supported by NSFC (No. 11401323, No. 11771272) and Jiangsu Government Scholarship for Overseas Studies.

Received March 21, 2017; Revised September 7, 2017

introduced by Buan et al. [4]. For a quiver $Q$ without loops and oriented 2-cycles and the corresponding path algebra $kQ$ (over an algebraically closed field $k$), the cluster category $\mathcal{C}_Q$ is the orbit category of the bounded derived category $D^b(kQ)$ by the functor $\tau^{-1}$ [1], where $\tau$ denotes the Auslander-Reiten translation and [1] is the shift function on the triangulated category $D^b(kQ)$.

Important objects in cluster categories are the cluster-tilting objects. The endomorphism algebras of such objects in the cluster category $\mathcal{C}_Q$ are called *cluster-tilted algebras* of type $Q$ [5]. Cluster-tilted algebras have several interesting properties, e.g., their representation theory can be completely understood in terms of the representation theory of the corresponding path algebra of a quiver (see [5]). These algebras have been studied by various authors, see for instance [1, 2, 7, 8].

In 2010, Yao [13] studied the properties of torsionless modules over cluster-tilted algebras of type $A_n$ and $D_n$. In this paper, we study the properties of torsionless modules over cluster-tilted algebras of type affine $A_n$. All the indecomposable torsionless modules are local and they are uniserial modules when they are not projective. The number of indecomposable nonisomorphism torsionless modules is only related to the number of arrows and oriented 3-cycles in the ordinary quivers and their relation expression is given.

### 2. The Ordinary Quivers of Cluster-tilted Algebras of Type Affine $A_n$

In this section, we give out the relations between the ordinary quivers of cluster-tilted algebras of type $A_n$ and affine $A_n$.

First, we recall the construction of the ordinary quivers of cluster-tilted algebras of type $A_n$: we can start with this two kinds of quivers $\bullet \longrightarrow \bullet$ and $\begin{tikzcd}[sep=small] \bullet \arrow[rr] & & \bullet \arrow[dl] \\ & \bullet \arrow[ul] & \end{tikzcd}$, whose vertices are called *free vertices*. Here, the composition of any two arrows in the 3-cycle is zero. Gluing two vertices of two quivers of the upper kinds together, we get a new quiver and a gluing point which is called a *nonfree vertex*. The free vertices of the new quiver can glue with a vertex of an arrow or a 3-cycle. Hence, we get a new quiver and a nonfree vertex too. Repeating this process, we can obtain all the ordinary quivers of cluster-tilted algebras of type $A_n$.

**Example 2.1.**

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[rr] & & \bullet \arrow[rr] \arrow[dl] & & \bullet \arrow[dr] & & \bullet \arrow[ll] \arrow[rr] & & \bullet \\
& \bullet \arrow[ul] & & & & \bullet \arrow[ur] \arrow[dr] & & & \\
& & & & \bullet \arrow[ur] \arrow[rr] & & \bullet & &
\end{tikzcd}$$

is an ordinary quiver of cluster-tilted algebra of type $A_n$; but

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[rr] & & \bullet \arrow[rr] \arrow[dl] & & \bullet \arrow[dr] & & \bullet \arrow[ll] & & \\
& \bullet \arrow[ul] & & & & \odot \arrow[ur] \arrow[dr] \arrow[rrr] & & & \bullet \\
& & & & \bullet \arrow[ur] & & \bullet \arrow[ll] & &
\end{tikzcd}$$

is not. Because after gluing with an oriented 3-cycle, the starting point of the arrow at $\odot$ is not a free vertex which cannot glue together with an oriented 3-cycle again.

In [3], Bastian gave the following definition. Let $\mathcal{Q}_n$ be the class of quivers with $n+1$ vertices which satisfy the following conditions:

(i) There exists precisely one full subquiver which is a non-oriented cycle of length $\geq 2$. Thus if the length is two, it is a double arrow.

(ii) For each arrow $x \xrightarrow{a} y$ in this non-oriented cycle, there may (or may not) be a vertex $z_a$ which is not on the non-oriented cycle, such that there is an oriented 3-cycle of the form

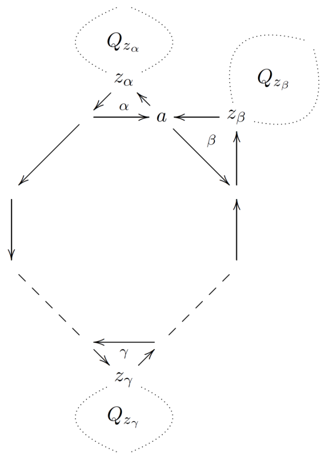
$$\begin{tikzcd}
& z_\alpha \arrow[dl] & \\
x \arrow[rr, "\alpha"] & & y \arrow[ul]
\end{tikzcd} .$$

Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the non-oriented cycle.

(iii) If we remove all vertices in the non-oriented cycle and their incident arrows, the result is a disconnected union of quivers $Q_1, Q_2, \ldots,$ one for each $z_\alpha$ (which we call $Q_\alpha$ in the following).

These are quivers of type $A_{k_a}$ for $k_\alpha \geq 1$, and the vertices $z_\alpha$ have at most two incident arrows in such a quiver, then $z_\alpha$ is a vertex in an oriented 3-cycle in $Q_\alpha$.

We can see the following figure for an illustration.

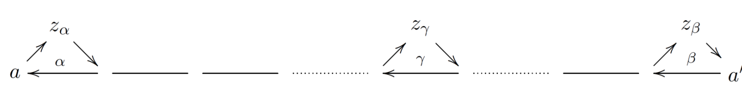



(2.1)

J. Bastian proved that $\mathcal{Q}_n$ are exactly the ordinary quivers of all cluster-tilted algebras of type affine $A_n$. For the convenience of narrative, the quiver will be called *reduced quiver*, if it has no $Q_{z_\alpha}$; and the non-oriented cycle will be called *center cycle*. We have the following lemmas:

**Lemma 2.2.** *Every ordinary quiver of a cluster-tilted algebra of type affine $A_n$ can become an ordinary quiver of a cluster-tilted algebra of type $A_n$ by disconnecting the non-oriented cycle.*

**Proof.** Without loss of generality, we suppose that the ordinary quiver of a cluster-tilted algebra of type affine $A_n$ like quiver (2.1). Disconnecting the quiver (2.1) from all the vertices $z_\alpha$, the new points also denoted by $z_\alpha$. By the condition (iii) in the definition of $\mathcal{Q}_n$, we know that all the vertices $z_\alpha$ are free in both $\mathcal{Q}_{z_\alpha}$ and the reduced quiver. Then disconnecting the reduced quiver from the vertex $a$, we get an ordinary quiver of a cluster-tilted algebra of type $A_n$:




, (2.2)

where the direction of "———" is just as that in the quiver (2.1). Gluing all the vertices $z_\alpha$ in the quiver $Q_{z_\alpha}$ together with all the vertices $z_\alpha$ in the quiver (2.2), then we can obtain an ordinary quiver of cluster-tilted algebra of type $A_n$:

$$\begin{tikzcd}[column sep=small]
 & Q_{z_\alpha} \arrow[d, dash, dotted] & & & & & Q_{z_\gamma} \arrow[d, dash, dotted] & & & & Q_{z_\beta} \arrow[d, dash, dotted] & \\
 & z_\alpha \arrow[dr] & & & & & z_\gamma \arrow[dr] & & & & z_\beta \arrow[dr] & \\
a \arrow[ur] & & \cdot \arrow[ll, "\alpha"] \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash, dotted] & \cdot \arrow[ur] & & \cdot \arrow[ll, "\gamma"] \arrow[r, dash, dotted] & \cdot \arrow[r, dash] & \cdot \arrow[ur] & & a' \arrow[ll, "\beta"]
\end{tikzcd} \quad . \tag{2.3}$$

□

**Lemma 2.3.** *Let i and j be two free vertices in an ordinary quiver Q of a cluster-tilted algebra of type* $A_n$ *satisfying the following conditions*:

(i) *The distance between i and j is longer than* 2 *in the underline diagram of the quiver* $Q$, *and*

(ii) *There is no nonzero path from i to* $j$.

*Then*, *we can get an ordinary quiver of cluster-tilted algebra of type affine* $A_n$ *by gluing i together with* $j$.

**Proof.** Suppose that the distance between $i$ and $j$ is $d \geq 2$. Then there is a subdiagram:

$$i \text{ ——— } i_1 \text{ ——— } i_2 \cdots\cdots \text{ ——— } i_{d-1} \text{ ——— } j\, , \tag{2.4}$$

where the direction of "——" is just as that in $Q$. By gluing $i$ together with $j$, we can get a non-oriented cycle whose length is $d \geq 2$. That is to say the condition (i) in the definition of $Q_n$ is satisfied.

Denoting by $i_0 = i$ and $i_d = j$, $i_k$, $i_{k+1}$, $i_{i+2}$ cannot be in a 3-cycle for any $0 \leq k \leq d-2$ by the definition of distance. By the constructions of the ordinary quivers of cluster-tilted algebras of type $A_n$, we know that the arrow in (2.4) may (may not) be in an oriented 3-cycle with the third vertex not in (2.4). Addition with all the vertices and incident arrows, we get a quiver like (2.2) which is a full subquiver of $Q$. So, gluing the vertices $i$ and $j$ together, we obtain a new quiver satisfying the condition (ii) in the definition $\mathcal{Q}_n$.

If every arrow $i_l$ —— $i_{l+1}$ for $0 \leq l \leq d-1$, is not in an oriented 3-cycle, then $Q$ is (2.4). So, we get an ordinary quiver of cluster-tilted algebras of type affine $A_n$

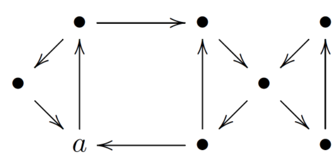
by gluing $i$ together with $j$. If there are some arrows in (2.4) in oriented 3-cycles, then we suppose the vertices which are not in (2.4) are $z_l$ for some $l$. Since $Q$ is an ordinary quiver of cluster-titled algebras of type $A_n$, there are only two cases:

(i) $z_l$ is free, or

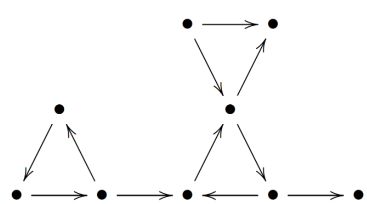
(ii) $z_l$ is not free which is connected with a quiver $Q_{z_l}$, where $Q_{z_l}$ is an ordinary quiver of cluster-tilted algebras of type $A_n$. In this case, there are at most two arrows in $Q_{z_l}$ connected with $z_l$. If there are two, then $z_l$ is a vertex of an oriented 3-cycle in $Q_{z_l}$.

So, we can get a new quiver which satisfies the condition (iii) in the definition of $\mathcal{Q}_n$ by gluing $i$ and $j$ together. This completes the proof. □

**Example 2.4.** Let $Q$ be the quiver




Then $Q$ is an ordinary quiver of cluster-tilted algebras of type affine $A_n$. If we disconnect the quiver $Q$ from the vertex $a$, then we get an ordinary quiver of cluster-tilted algebras of type $A_n$:



### 3. The Number of Indecomposable Noisomorphism Torsionless Modules

Let $\operatorname{indSub}(A)$ be the set of all the indecomposable nonisomorphism torsionless modules over an algebra $A$. $\operatorname{Sub}(A) = \{ {}_A X \mid X$ is a torsionless module$\}$;

$\underline{\mathrm{Sub}}(A) = \mathrm{Sub}(A)/\mathrm{Proj}(A)$, where $\mathrm{Proj}(A)$ denotes all the projective $A$-modules. In [13], Yao has described the torsionless modules over cluster-tilted algebras of type $A_n$ and their number. In order to describe the number of torsionless modules of a cluster-tilted algebra of type affine $A_n$, we should recall the following three lemmas in [13]:

**Lemma 3.1.** *Let A be a finite dimensional k-algebra. Then B is the path algebra of the following quiver* $Q'$:

$$0 \xleftarrow{\alpha} 1 \quad Q \quad ,$$

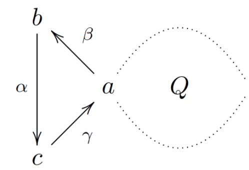
*where* $Q$ *is the ordinary quiver of the algebra* $A$. *Here the arrow* $\alpha : 1 \to 0$ *is not involved in any relations. Then there is a bijection between* $\mathrm{indSub}(A)$ *and* $\mathrm{indSub}(A)/S(0)$, *where* $S(0)$ *is the simple* $B$-*modules corresponding to the vertex* $0$. *Hence there is a bijection between* $\mathrm{ind}\underline{\mathrm{Sub}}(A)$ *and* $\mathrm{ind}\underline{\mathrm{Sub}}(B)$.

**Lemma 3.2.** *Let A be a finite dimensional k-algebra. Then B is the path algebra of the following quiver* $Q'$:

$$0 \xrightarrow{\alpha} 1 \quad Q \quad ,$$

*where* $Q$ *is the ordinary quiver of the algebra* $A$. *Here the arrow* $\alpha : 1 \to 0$ *is not involved in any relations. Then there is a bijection between* $\mathrm{indSub}(A)$ *and* $\mathrm{indSub}(A)/P(0)$, *where* $P(0)$ *is the indecomposable projective* $B$-*modules corresponding to the vertex* $0$. *Hence there is a bijection between* $\mathrm{ind}\underline{\mathrm{Sub}}(A)$ *and* $\mathrm{ind}\underline{\mathrm{Sub}}(B)$.

**Lemma 3.3.** *Let A be a finite dimensional k-algebra. Then B is the path algebra of the following quiver* $Q'$:



*where $Q$ is the ordinary quiver of the algebra $A$. Here the arrow $\beta : a \to b$ and $\gamma : c \to a$ are not involved in any relations except $\alpha\beta = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$. Then there is a bijection between $\mathrm{indSub}(B)/\{P(b), P(c), S(b), S(c), \mathrm{rad}P(c)\}$ and $\mathrm{indSub}(A)$ with $P(b)$, $P(c)$ are the indecomposable projective $B$-modules and $S(b)$, $S(c)$ the simple $B$-modules corresponding to the vertices $b$ and $c$. Hence there is a bijection between $\mathrm{ind}\underline{\mathrm{Sub}}(B)/\{S(b), S(c), \mathrm{rad}P(c)\}$ and $\mathrm{ind}\underline{\mathrm{Sub}}(A)$.*

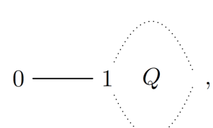
**Lemma 3.4.** *Let $Q$ be an ordinary quiver of cluster-tilted algebras of type $A_n$, where $Q$ consists of $s$ arrows and $t$ oriented 3-cycles. Then the number of all the indecomposable nonisomorphism torsionless modules is $l(A) = s + 5t + 1$.*

**Proof.** Using induction on the number of non-free vertices in $Q$. If $Q$ has no non-free vertex, then $Q$ only can be the following two cases:

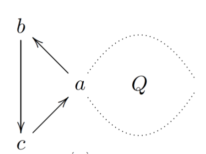
$$\begin{tikzcd} & c \arrow[dl] & \\ a \arrow[rr] & & b \arrow[ul] \end{tikzcd} \qquad l(A) = 6,$$

$$a \longrightarrow b \qquad l(A) = 2.$$

In both cases, the lemma is true. Suppose that the lemma is true whenever $Q$ has $m$ non-free vertices, i.e., $l(A) = s + 5t + 1$, where $s$ and $t$ are the numbers of arrows and oriented 3-cycles in $Q$, respectively. Now, we consider the cluster-tilted algebras $B$ of type $A_n$ whose ordinary quiver is $Q'$ with $m + 1$ non-free vertices. Since $Q'$ can be obtained by adding an arrow or an oriented 3-cycle to a quiver $Q$ with $m$ non-free vertices. Hence $Q'$ only can be the following two cases:



(1)

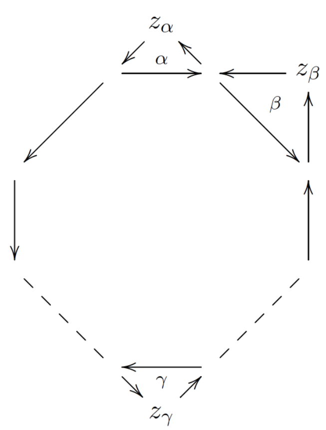


(2)

For the case (1), we have $l(B) = (s+1) + 5t + 1$ by Lemmas 3.1 and 3.2; and (2), we have $l(B) = s + 5(t+1) + 1$ by Lemma 3.3. So the lemma is true. □

**Remark 3.5.** In [13], Yao gave a formula $l(A) = 6c - d + e$, where $A$ is a cluster-tilted algebra of type $A_n$, $c$ is the number of oriented 3-cycles in the ordinary quiver of $A$, $d$ is the number of connecting vertices and $e$ is the number of formal vertices. By the above Lemma 3.4, we have $e + c - d - s = 1$. This is some like the Euler formula.

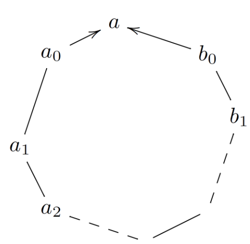
By the above three Lemmas 3.1-3.3 and the construction of $Q_{z_\alpha}$, in order to consider the number of torsionless modules of cluster-tilted algebras of type affine $A_n$, we just need to consider those of the quivers without any $Q_{z_\alpha}$, i.e.,



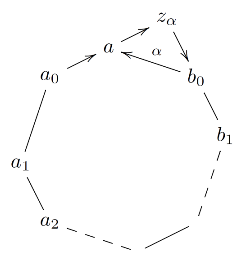

(2.5)

**Lemma 3.6.** *Let $Q$ be an ordinary quiver of cluster-tilted algebras of type affine $A_n$. Here $Q$ is a reduced quiver which has no $Q_{z_\alpha}$. Then the number of all the indecomposable nonisomorphism torsionless modules is $l(A) = 5t + s$, where $s$ and $t$ are the numbers of arrows and oriented 3-cycles in $Q$, respectively.*

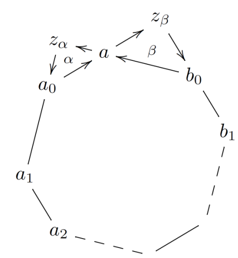
**Proof.** Since $Q$ has a non-oriented cycle, there is at lest one vertex $a$ which is a sink vertex in the non-oriented cycle as a subquiver of $Q$. By the definition of $\mathcal{Q}_n$, the arrows in the non-oriented cycle may (or may not) be in an oriented 3-cycle, then there are three cases at vertex $a$:




(1)



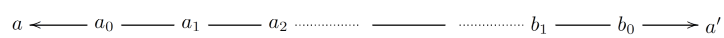

(2)



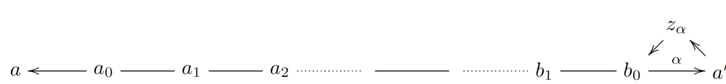

(3)

where "——" stands for an arrow whose direction is arbitrary and may (or may not) be in an oriented 3-cycle.

Disconnecting the above quivers at vertex $a$ and denoting the two new vertices by $a$ and $a'$, respectively, we can get the following three ordinary quivers $Q'$ of cluster-tilted algebras $B$ of type $A_n$ correspondingly:




(1′)




(2′)


(3′)

We just need to prove that $l(A) = l(B) - 1$ in all the three cases. The proofs of all the three cases are similarly. So in the following we only prove Case (3). Let $p_1 : a'_k \to \cdots \to a'_1 \to a_0 \to a \to z_\beta$ and $p_2 : b'_m \to \cdots \to b'_1 \to b_0 \to a \to z_\alpha$ be the longest nonzero paths through the vertex $a$ in the quiver (3), then $p'_1 : a'_k \to \cdots \to a'_1 \to a_0 \to a$ and $p'_2 : b'_m \to \cdots \to b'_1 \to b_0 \to a'$ be the longest nonzero paths through the vertex $a$ and $a'$ in the quiver $(3')$, respectively. Since $Q$ is a reduced quiver, a nonzero path pass through a either a subpath of $p_1$ or a subpath of $p_2$. Let the set $I = \{a'_k, \ldots, a'_1, a_0, a, b'_m, \ldots, b'_1, b_0\}$, $\mathrm{indSub}(A, I) = \{X \in \mathrm{indSub}(A) \mid X \subseteq P(i)$ for some $i \in I\}$ and $\mathrm{indSub}(B, J) = \{X \in \mathrm{indSub}(B) \mid X \subseteq P(j)$ for some $j \in J \subseteq Q'_0\}$. Then by the construction of the quiver (3) and the quiver $(3')$, we know that $\mathrm{indSub}(A) = \mathrm{indSub}(A, I) \bigcup \mathrm{indSub}(A, Q_0 \setminus I)$ and $\mathrm{indSub}(B) = \mathrm{indSub}(B, I \bigcup \{a'\}) \bigcup \mathrm{indSub}(B, Q_0 \setminus (I \bigcup \{a'\}))$. By the relation between (3) and $(3')$, we can get $\sharp\mathrm{indSub}(A, Q_0 \setminus I) = \sharp\mathrm{indSub}(B, Q_0 \setminus (I \cup \{a'\}))$ and $\sharp\mathrm{indSub}(A, I) = t + s + 5$, $\sharp\mathrm{indSub}(B, I \cup \{a'\}) = t + s + 6$ by the properties of indecomposable projective modules and their submodules. Hence $l(A) = l(B) - 1$. By Lemma 3.4, we get $l(A) = 5t + s$. This completes the proof.

$\square$

## 4. The Properties of Torsionless Modules

In this section, we want to discuss the properties of torsionless modules over cluster-tilted algebras of type affine $A_n$.

**Lemma 4.1.** *Let $Q = (Q_0, Q_1)$ be an ordinary quiver of cluster-tilted algebras of type $A_n$ and $i \in Q_0$ be a free vertex. Then there is at most one maximal nonzero path starting at $i$, i.e., each nonzero path starting at i is its subpath.*

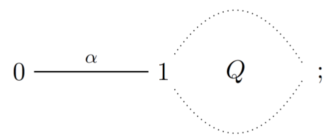
**Proof.** Using induction on the number *m* of nonfree vertices.

If $m = 0$, then *Q* only can be the following two cases:

$\bullet \longrightarrow \bullet$ and
$$\begin{tikzcd} \bullet \arrow[rr] & & \bullet \arrow[dl] \\ & \bullet \arrow[ul] & \end{tikzcd}$$
. It is clear that the lemma is true in this case.

Suppose the lemma is true when *Q* has *m* nonfree vertices. In the following, we need to prove that the lemma is true for $Q'$ having $m + 1$ nonfree vertices. Since $Q'$ can be obtained from *Q* by adding an arrow or an oriented 3-cycle, $Q'$ may be one of the following two case:

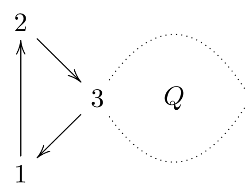
(1)




The vertex 1 is free before adding $\alpha$ to $Q$. By the inductive hypothesis, there is only one maximal nonzero path starting at 1. After adding $\alpha$ to $Q$, 1 becomes a nonfree vertex and 0 is a new free vertex of $Q'$. If $\alpha$ is from 0 to 1, then the nonzero paths starting at 0 are just the composition of $\alpha$ and those starting at 1 in $Q$. Hence there is only one maximal nonzero path starting at 0. If $\alpha$ is from 1 to 0, no path starts at 0.

For a free vertex $i$ in $Q$, if no nonzero path starting at $i$ passes through 1, then the nonzero paths starting at $i$ in $Q'$ are just as those in $Q$; if not, the nonzero paths starting at $i$ passing through 1 in $Q'$ are just as those in $Q$ or extended to 0. So there is only one maximal nonzero path starting at a free vertex in $Q'$.

(2)




Before adding the oriented 3-cycle to $Q$, the vertex 3 is free. By inductive hypothesis, there is only one maximal nonzero path starting at 3 in $Q$. From the above quiver, we know that there is only one nonzero path $1 \to 2$ starting at 1. The nonzero paths starting at 2 are the compositions of $2 \to 3$ and nonzero paths starting at 3 in $Q$. Similarly in Case (1), we can prove that there is only one maximal nonzero path in $Q'$. □

**Lemma 4.2.** *Let $Q = (Q_0, Q_1)$ be an ordinary quiver of cluster-tilted algebras of type $A_n$. Then the nonzero paths starting at each vertex $i \in Q_0$ only can be the following two cases*:

(1) $i \to b_1 \to \cdots \to b_t$,

(2) $a_s \leftarrow \cdots \leftarrow a_1 \leftarrow i \to b_1 \to \cdots \to b_t$,

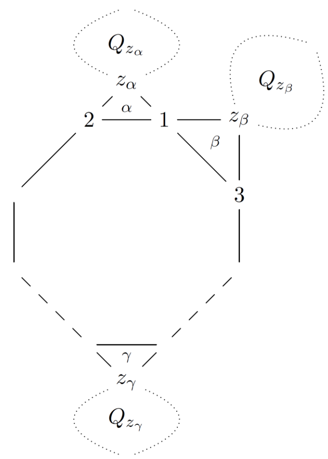
*where* $a_s \leftarrow \cdots \leftarrow a_1 \leftarrow i$ *and* $i \to b_1 \to \cdots \to b_t$ *are maximal and* $a_i \neq b_j$, $\forall i \in Q_0$, $\forall j \in Q_0$.

**Proof.** If $i$ is a free vertex, then the assertion is true by Lemma 4.1. If $i$ is a nonfree vertex, then we obtain two ordinary quivers $Q'$ and $Q''$ of cluster-tilted algebras of type $A_n$ with $i$ being a free vertex in both quivers by disconnecting $Q$ at $i$. By Lemma 4.1, there is only one nonzero path starting at $i$ in $Q'$, denoting it by $a_s \leftarrow \cdots \leftarrow a_1 \leftarrow i$. Similarly, there is only one nonzero path starting at $i$ in $Q''$, denoting it by $i \to b_1 \to \cdots \to b_t$. So, the nonzero paths starting at $i$ is just like: as $a_s \leftarrow \cdots a_1 \leftarrow i \to b_1 \to \cdots \to b_t$, where $a_i \neq b_j$, $\forall i \in Q_0$, $\forall j \in Q_0$. This completes the proof. □

Using the above lemmas, we have some corollaries which have been given in [13]:

**Corollary 4.3.** *Let M be an indecomposable torsionless module over cluster-tilted algebras A of type* $A_n$. *Then M is local. If M is not projective, then it is a uniserial module.*

**Proof.** By Lemma 4.2, for any $i$ in the ordinary quiver of $A$, the indecomposable projective $P(i)$ accordingly to $i$ is either $k \to k \to \cdots \to k$, or $k \leftarrow \cdots \leftarrow k \leftarrow k \leftarrow k \to \cdots k$. So $P(i)$ is a local module and its indecomposable proper submodules are uniserial modules. This completes the proof. □




(2.6)

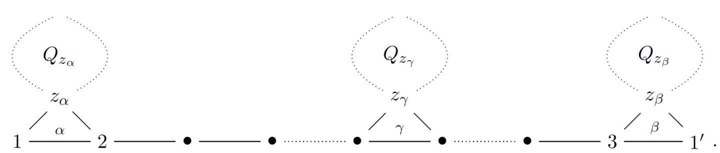
**Lemma 4.4.** *Let* $Q = (Q_0, Q_1)$ *be an ordinary quiver of cluster-tilted algebras of type affine* $A_n$. *Then the nonzero paths starting at each vertex* $i \in Q_0$ *only can be the following two cases*:

(1) $i \to b_1 \to \cdots \to b_t$,

(2) $a_s \leftarrow \cdots \leftarrow a_1 \leftarrow i \to b_1 \to \cdots \to b_t$,

*where* $a_s \leftarrow \cdots \leftarrow a_1 \leftarrow i$ *and* $i \to b_1 \to \cdots \to b_t$ *are maximal and there exists at most one i and one j such that* $a_i = b_j$.

**Proof.** Without loss of generality, we can suppose that $Q = (Q_0, Q_1)$ is the quiver (2.6), where the direction of "——" is arbitrary,  stands for an oriented 3-cycle. Disconnecting the quiver $Q$ from the vertex 1, denoting the new vertices by 1, $1'$, respectively, we can get the following ordinary quiver $Q'$ of cluster-tilted algebras of type $A_n$:




If all the paths starting at $i$ in $Q$ do not pass through the vertex 1, then those starting at $i$ in $Q'$ are the same as in $Q$. By Lemma 4.2, we know that the assertion is true at this case.

If there is a path starting at $i$ in $Q$ passing through the vertex 1, we discuss this occasion in the following two cases:

(i) There is a vertex $j$ in the non-oriented cycle, such that all the nonzero paths starting at $i$ do not pass through $j$. Then we disconnect $Q$ at $j$ to get an ordinary quiver $Q'$ of cluster-tilted algebras of type $A_n$. So, all the nonzero paths starting at $i$ in $Q$ are the same as those in $Q'$.

By Lemma 4.2, the assertion is true at this case.

(ii) All the vertices in the non-oriented cycle are in the nonzero paths starting at $i$. At this case, there is only one sink vertex $m$ and only one source vertex $i$ in the non-oriented cycle. At this time, disconnecting $Q$ at $i$, we get a new ordinary quiver $Q'$ of cluster-tilted algebras of type $A_n$ and two new vertices denoting by $i$ and $i'$ which are free vertices in $Q'$. By Lemma 4.1, there is only one maximal nonzero path starting at $i$ and $i'$, respectively and $m$ is the unique intersection of the two maximal nonzero path. Gluing $i$ and $i'$ together, the nonzero path starting at $i$ in $Q$ is just like Case 2. This completes the proof. □

**Theorem 4.5.** *Let $M$ be an indecomposable torsionless module over cluster-tilted algebras of type affine $A_n$, then $M$ is local. If $M$ is not projective, then it is a uniserial module.*

**Proof.** It follows from Lemma 4.4 directly. □

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